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AUTHOR: Ronald Harold Rasch

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STATEMENT(s):

VITA

Ronald Harold Rasch [redacted] He graduated from high school in Buffalo in 1959. In 1966 he entered Kansas State University through the Airman Education and Commissioning Program. He received the degree of Bachelor of Science, cum laude, in Nuclear Engineering in 1969 and was elected to Sigma Tau. He attended Officer Training School where he was designated as a distinguished graduate and was commissioned in the USAF in May, 1969. He received the degree of Master of Science, with distinction, with a major in Systems Management from the Air Force Institute of Technology in 1970. Following graduation he was employed as a system analyst responsible for the development and implementation of management information systems for Air Force research and development projects. In 1975 he joined the faculty at the United States Air Force Academy. In August, 1977 he entered the Graduate School of The University of Texas at Austin.

[redacted]

This dissertation was typed by Phyllis Ann Reynolds.

AN ANALYSIS OF INFORMATION STRUCTURE AND OPTIMAL TRANSFER PRICING DECISION RULES FOR DECENTRALIZED FIRMS

by

RONALD HAROLD RASCH, B.S., M.S.

DISSERTATION

Presented to the Faculty of the Graduate School of

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Of course, despite the contributions from many people, I take the full responsibility for errors that may remain.

— Ronald H. Rasch

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FOR DECENTRALIZED FIRMS

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Chapter III introduces the concept of linear quadratic control and develops a decentralized model of the firm. We develop the concepts of decentralized decision making and decentralized information availability to formulate a well posed problem of decentralized control.

The solution of the decentralized control problem is presented in Chapter IV where we derive an optimal decentralized control policy for a general organizational team using the mathematical approach of dynamic programming and the results of team theory. This policy is then applied to our research problem to determine the effect of transfer pricing policy on the decentralized decision maker's actions. The main finding of this study under the stated conditions of the analysis is that the transfer price involved in the interdivisional exchange of goods or services does not affect the decentralized team decision maker's actions.

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CHAPTER I

INTRODUCTION AND BACKGROUND

Introduction

The rapid rate of technological advancements and a steadily increasing growth in the size of business organizations have resulted in a trend toward divisionalization of these organizations in recent years. Divisionalization leads to the encouragement of creative talents of responsive individuals, a readily available measure of segment success in the form of profit contribution and the improvement of management training (Solomons: 1965). Decentralized decision making results in separate divisions which are essentially autonomous profit centers. In a world of certainty the objectives of decentralization in business organizations through the creation of profit centers are likely to be achieved when the managers of various profit centers, acting in their own self-interest, also maximize central management's preferences. This idea holds trivially when there are no interactions between the various decision centers; i.e., no flow of goods or services between decision centers and no cost or demand interdependence (Kanodia:1979). However, these

divisions are frequently faced with the problem of pricing goods and services that they exchange with each other. Internal transfer prices for these goods and services are historically derived from a corporate headquarters or central coordination agency or through divisional negotiation. The problem of establishing these prices is important since they affect divisional goal congruence, individual incentive and autonomy of decision making (Horngren:1977).

Statement of the Problem

Classical microeconomic theory assumes that the centralized decision maker has perfect knowledge of all the information required for decision making. However, we observe that actual decisions are based on information that is both incomplete and imperfect due to the uncertain environment that the firm must operate within. Yet the formal admission of uncertainty has not been acknowledged in most of the transfer pricing literature to date (Demski:1975). This admission is important because the results obtained under subjective certainty do not necessarily extend to an uncertain setting. In addition, questions relating to information discrepancies, communication strategies and risk sharing only arise in an uncertain setting.

Another premise of classical economic theory is that the firm operates in a static environment where current decisions do not impact future periods of operation. In reality, the decision maker must base these decisions on a changing environment where the information concerning the environment is also changing over time. A dynamic analysis of this information structure has not been addressed in the transfer pricing literature to date. Furthermore, the complex problem of decentralized decision making under conditions of limited informational structures in an uncertain environment has not been acknowledged in the literature to date.[1]

The problem of establishing an optimal (in some sense) transfer pricing policy for a decentralized firm in an uncertain, dynamic environment provides the impetus for this dissertation. In the above sentence, the word "optimal" refers to the development of a control policy which will encourage the maximization of a predetermined measure of profit. The purpose of the control system is not to explicitly develop organizational planning objectives, although the feedback received from a viable control system frequently results in planning adjustments. This type of closed loop system meshes the functions of

[1]Important exceptions are Arrow (1964), Groves (1973), Marschak (1959) and Wilson (1968).

management planning and control in actual operating circumstances; however, for the purpose of this dissertation, it is assumed that optimal control refers to the actions required to optimize predetermined planning objectives.

Background

In an organization, individuals normally differ in at least three important aspects: (1) they control different action variables; (2) they base their decisions on different information; and (3) they have different preferences; i.e., tastes and beliefs. A normative analysis of organizations could thus be suitably modelled as a methematical game theory problem (Radner:1972a). However, many interesting aspects of organizations are related to differences of types (1) and (2) only. Furthermore, in some cases the members of an organization may have nearly identical preferences. Finally, in the present state of development, the theory of games of more than two persons does not appear to provide many clues as to how to proceed in a general analysis of organizations (Radner:1972b). This suggests the study of theoretical organizations in which differences of type (3) are absent; that is, in which preference differences are neglected and a single payoff function reflects the common goals of the members. Jacob Marschak (1955) has termed such an organization a team. In the theory of teams two basic

questions are investigated: (1) for a given information structure, what is the optimal decision function; and (2) what are the relative values of alternative information structures.

The impact of a transfer pricing change on the actions of decentralized decision makers has not been investigated in a team setting. This impact may not be trivial in that it is not clear exactly how decentralized decision makers evaluate available information. This dissertation research will attempt to develop a decentralized control model of the firm that will enable us to evaluate the impact of various information patterns, to include the transfer price, with respect to the actions taken by the decentralized decision makers. The dissertation will employ a team theoretic approach to the decentralized control problem which will allow investigation of the impact of a change in transfer pricing policy on the decision maker's actions. Most of the team theory literature has not addressed the team problem in a dynamic environment; however, current research in the fields of economics and engineering has begun to deal effectively with dynamic models through the use of modern control theory analysis.

The mathematical foundations of certain parts of modern control theory can be traced back to works that

were completed some seventy years ago. For instance, the state variable approach to linear systems is well known to mathematicians as the theory of first-order linear differential equation solutions. State space concepts, fundamental to modern control theory, evolved from the classical theory of dynamics of particles and rigid bodies, referred to as phase-space (Fuller:1960). One of the significant aspects of modern control theory is that it is useful in the analysis of multivariate, stochastic, dynamic systems. Until recently, a modern control theoretic approach was limited to engineering problems dealing with the physical sciences. This approach has now received attention in various fields of the social sciences, particularly in economic research. The apparent widespread use of modern control theory techniques to economic research can be summarized as follows:

> Control theory methods are used to find the optimal set of policies over time to direct a deterministic system or stochastic system from given initial conditions to desired terminal conditions. Since a large number of economic problems are naturally described as dynamic systems which can be influenced by policies in an attempt to improve their performance, control theory has gained widespread application by economists. (Kendrick:1980)

Objective of Research

The objective of this dissertation is to develop a conceptual framework for the analysis and control of decentralized decision making for a firm operating in a

dynamic environment under conditions of uncertainty. The framework will incorporate a modern control theoretic mathematical structure and employ a team theoretic approach to the analysis of organizational behavior. Thus the conceptual framework will attempt to embody the economic concepts of team theory with the mathematical concepts of modern control theory to analyze the dynamic problem of optimal information structure and transfer pricing policy which is of interest to the accountant.

Past Approaches and the Approach of This Study

Early accounting research concentrated on a pragmatic approach to the transfer pricing problem.[2] Classical economic analysis of the transfer pricing problem was conducted by Hirshleifer (1956) and his paper is the definitive reference in the literature. However, Hirshleifer's procedure requires complete knowledge of market situations and complete communication between decision makers; conditions that rarely, if ever, exist in the current business environment.

Mathematical programming approaches to the establishment of transfer prices were presented by Baumol and Fabian (1964), Jennergren (1972) and Bailey and Boe (1976).

---

[2] See, for example, Cook (1955), Dean (1955), Dearden (1964) and Stone (1956).

These approaches require time-consuming iterations of information exchanges that are based on sensitive optimality assumptions. The transfer pricing literature was surveyed by Abdel-Khalik and Lusk (1974) and they conclude that the above approaches have produced more questions than answers.

The approach of this dissertation acknowledges the uncertain and dynamic environment that exists in a modern decentralized organization and employs a modern control theoretic team approach to decentralized decision making. Our conceptual framework operates in a dynamic environment, incorporates conditions of uncertainty, allows for multiple information structures and addresses the issue of pricing interdivisional transfers of goods and services. Decision making for the divisions resides with the respective division managers (decentralized decision making). Decentralized decision making refers to the following: given m decisions or actions to be made by n decision makers ($1<n\leq m$), each decision maker is assigned a subset of the m decisions. For the overall system there is a given criterion function and a space of possible choices involving the m decisions. Each decision maker is assigned a space of possible choices and a criterion function involving at least the decision

variables which he can partially or totally control (Whinston:1964).

The information set available to each decision maker may also vary. Suppose that the ith person observes a random variable $y_i(x)$ and takes action a_i. If there is no communication among the persons, then person i's information function is defined as $I_i(x) = y_i(x)$. However, if there is complete communication among all n persons, then $I_i(x) = Y(x) = (y_1(x), y_2(x),\ldots, y_n(x))$. Rarely does one encounter these two extremes of no communication or complete information in a real organization. Rather, we find that numerous devices are used to bring about a partial exchange of information (Radner: 1961).

Overview of Contents of the Dissertation

Chapter II of this dissertation provides an introduction to the concepts of modern control theory. We develop a classical model of the firm and demonstrate how modern control theory techniques are applicable to the dynamic optimization problem of the firm. The transition from static optimization to dynamic optimization theory is accomplished by reviewing the discrete time minimum principle and applying this principle to the classical problem of profit maximization.

Chapter III introduces the concept of linear quadratic control and develops a decentralized model of the firm. We develop the concepts of decentralized decision making and decentralized information availability to formulate a well posed problem of decentralized control.

The solution of the decentralized control problem is presented in Chapter IV where we derive an optimal decentralized control policy for a general organizational team using the mathematical approach of dynamic programming and the results of team theory developed by Radner (1962). This policy is then applied to our research problem to determine the effect of transfer pricing policy on the decentralized decision maker's actions.

CHAPTER II

CENTRALIZED MODEL OF THE FIRM

Introduction

As we have emphasized in the previous chapter, the purpose of this dissertation is to develop an analytic framework which will enable us to evaluate the performance of a decentralized firm operating in a dynamic environment under conditions of uncertainty. This chapter will provide the analytic background necessary to understand the inherent difficulties encountered when we extend the classical economic model of a firm to achieve the dissertation objective. A second purpose of the chapter is to develop an orderly transition from classical static optimization techniques to the modern control theory approach used to solve optimal control problems.

We will develop a static model of the firm and solve the attendant optimization problem. Next we will extend the model to a dynamic environment and develop a maximization principle that will enable us to solve the dynamic optimal control problem. Following chapters extend the model to incorporate decentralized information and decision making and address future uncertainties which

the firm faces and we develop a procedure to solve the resultant stochastic control problem.

Static Model of the Firm

Traditional management models of the firm normally include an organizational structure which incorporates the functions of production and marketing with related costs of distribution and production and related sales revenue. The firm's objective is normally that of profit maximization. The model we develop will include these concepts where profit becomes a function of both sales and costs. It should be noted that certain empirical evidence exists to suggest that firms may not regard profit maximization as their sole overriding objective.[3] The model we will develop does not explicitly require the assumption of profit maximization and could readily be extended to incorporate behavioral preferences through the recognition of a utility function which incorporates individual beliefs and preferences. For clarity of exposition our model will retain the profit maximization objective which implicitly assumes that the firm's utility function is linear in dollars. This assumption does not cause any conceptual difficulty for a centralized firm;

[3]See, for example Coase (1972), Cyert and March (1968), Jensen and Meckling (1976) and Salamon and Smith (1979).

however, it does become restrictive when we address decentralized decision making and will be discussed further at that point.

Figure 2.1 shows the eseential framework of the model we will develop. The functions of production (Division P) and marketing (Division M) are aligned under the control of a centralized decision maker (Headquarters). Raw material required for production is obtained in a perfectly competitive market where Division P can purchase any amount of material, q_r, at the prevailing market price, p_r. This amount is processed into a finished product, q_p. For ease of exposition we assume a single input production process with no internal loss. This notation assumes that the units of raw material and the final product are the same. This assumption could be relaxed by defining the amount of raw material purchased as αq_p where α represents the units of input required per production unit. These assumptions are not essential to the development of the model; however, the relaxation of them would unduly complicate the mathematics involved without adding additional clarity of exposition.

The marketing division receives the finished good q_p and distributes an amount q_f at a price p_f. Unlike the raw material market, the demand for the firm's product is a function of the sales price, p_f, which the firm

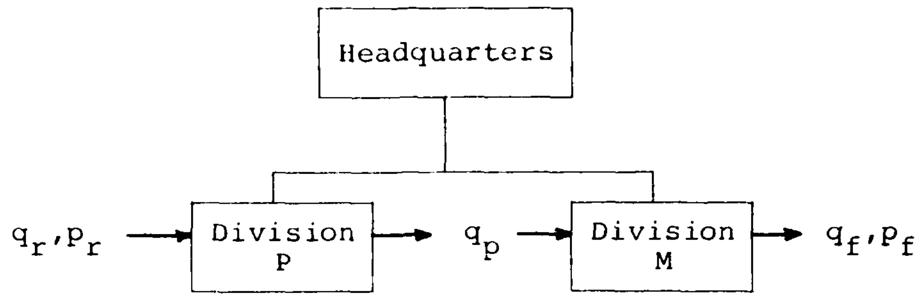

Fig. 2.1. Centralized Model of the Firm

must establish. To achieve as much realism as possible we will assume that the firm can mildly affect the demand for its product through advertising, product differentiation and price. Thus its market is neither purely competitive nor purely monopolistic. Chamberlin (1962) has termed such a situation as monopolistic competition and the interested reader is referred to his work for an extensive treatment of the subject.

We will assume that the firm has full knowledge of the demand function for its product which exhibits the relationship

$$q_f = -b_1 p_f + b_2$$

where $b_1 \geq 0$ and $b_2 \geq 0$ are known constants.

The firm's revenue is a function of sales quantity and sales price as

$$R = R(q_f, p_f) = p_f q_f.$$

The firm's cost function consists of production costs and marketing costs. We assume that the marketing division incurs a fixed cost, C_M, based on a fixed sales force and advertising budget. The production costs are a function of raw material costs and internal processing (labor and machine) costs. In economics we often encounter a production cost-volume relationship such as that given in Figure 2.2.

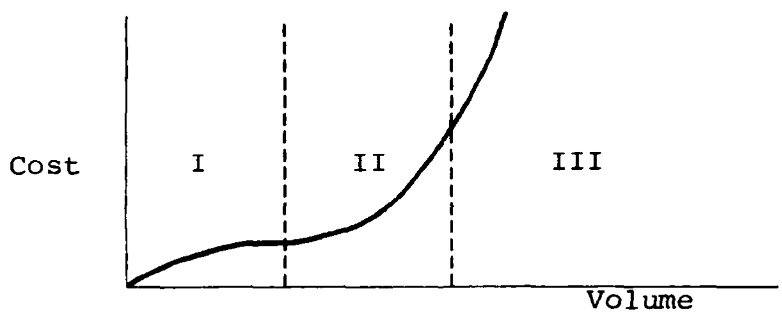

Fig. 2.2. Production Cost Function

Such a function has the property that marginal costs decrease initially but increase above a certain volume. Empirical evidence supports this relationship due to a learning effect and the improvement in efficiency that accompanies a volume increase (Itami:1973). However, we note that the majority of accounting cost analyses assume a linear production cost relationship of the form $c = av + b$ where c is total cost, v is total volume and a,b are known constants. Although this function is convenient for analytic purposes, we feel that

marginal cost must eventually rise as volume increases which is a property not incorporated using a linear cost function. To incorporate the concept of increasing marginal cost we will assume a quadratic cost function of the form $c = a + bv + cv^2$ where a,b,c are constants. This will enable us to preserve analytic tractability in our model and at the same time achieve an implicit objective of the analysis which is to capture as much realism as possible without unnecessarily complicating the model. By using a second-order approximation for production costs we assume that the firm is operating somewhere in region II (Figure 2.2). Classical microeconomic theory tells us that it is not efficient for a firm to operate in regions I or III and thus we feel that a second-order estimate of production cost is justified from both an empirical and theoretical perspective.[4] The quadratic cost function captures the concept of increasing marginal cost inherent in many organizations; however, it should be noted that many major industrial processes may not exhibit quadratic cost behavior and the application of this model may not be appropriate for firms that are not operating in region II.

---

[4]For a further discussion of quadratic cost curves and their fit to empirical data, see Holt, et al. (1960) and Spencer and Siegelman (1964).

The production cost is of the form

$$C_p = a_1 + p_r q_r + a_2 q_r^2$$

where a_1, a_2 are known constants and p_r is the unit cost of the raw materials. Thus a_1 represents fixed operating costs and a_2 captures the concept of increasing marginal costs while p_r captures the material component of the production costs.

The total profit of the firm becomes

$$\text{Profit} = P = \text{Revenue} - \text{Expenses}$$
$$P = R - C_p - C_M$$
$$P = p_f q_f - a_1 - p_r q_r - a_2 q_r^2 - C_M.$$

The firm's problem becomes:

$$\begin{aligned} &\text{Maximize: } P = p_f q_f - C_M - a_1 - p_r q_r - a_2 q_r^2 \\ &\text{Subject to: } q_f = -b_1 p_f + b_2. \end{aligned} \qquad \textbf{(A)}$$

It is interesting to observe at this point in the model's development that perfect information is assumed and thus the issue of optimal information structure is imbedded in the model's assumptions. Similarly, there is no transfer pricing problem at this juncture due to the centralized aspect of the model. From this perspective the model is currently uninteresting to the accountant concerned with the development of information systems and optimal transfer pricing policy. Following sections will

extend the model to allow us access to the issues of information structures and transfer pricing policy along with an appreciation of the difficulties encountered as we extend the model. This extension will involve a transition from classical static optimization techniques to optimal control theory techniques. To facilitate this transition, the next section will review classical optimization. We will then extend the model beyond the static case and develop the tools of optimal control theory which will enable us to pursue the analysis in the remainder of our research.

Static Optimization

Under the specified conditions of certainty problem A reduces to the following, which we designate problem B:

$$\text{Maximize: } p_f q_r - (C_M + a_1) - p_r q_r - a_2 q_r^2 \quad \text{(B-1)}$$

$$\text{Subject to: } q_r = -b_1 p_f + b_2. \quad \text{(B-2)}$$

Problem B can be solved as a static optimization problem where the firm's decision involves determining the amount of product produced and its price. Let us redefine this amount as d, where d represents the firm's production decision (note that $d = q_r = q_p = q_f$).

Substituting (B2) into (B1) yields the profit function

$$P = ((b_2-d)/b_1)d - p_r d - a_2 d^2 - (C_M+a_1)$$

then the first-order condition for the profit function is

$$dP/dd = (b_2-d)/b_1 - d/b_1 - p_r - 2a_2 d$$

denoting d* as the optimal decision:

$$d^* = (1/(2(1+b_1a_2)))\ (b_2-b_1p_r) \tag{2-1}$$

and the optimal sales price:

$$p_f^* = (b_2-d^*)/b_1 \tag{2-2}$$

Thus the firm will purchase d* units of raw material at price p_r and sell d* units of finished product at a price p_f^*. It is apparent that these results readily reduce to the classical economic result that the firm should price its product such that marginal revenue equals marginal cost.

This section has developed the results needed to solve the problem we formulated as problem B. However, the basic model we developed assumes that the firm's decision problem is not dynamic in that future changes in price, cost and the market demand were not allowed. In the next section we will extend the model to allow for a

dynamic, changing environment and begin to develop the analytic tools required to evaluate dynamic optimization problems using the techniques of optimal control theory.

Dynamic Model of the Firm

In this section we relax the demand and supply market assumptions inherent in problem A. The model will be extended to allow for both a changing environment with respect to demand for the firm's product and also a changing price in the raw material market. As a result of price instability in the world over the last ten years and no indication as to a reversal of those trends we assume that the price of raw material, p_r, will tend to rise over time. If we consider a discrete time period, t, we can represent this change as

$$p_r(t+1) = p_r(t) + kp_r(t)$$

where the argument represents the time period of interest. We no longer consider p_r to be constant, but rather a variable which is a function of time where the constant k represents the rate at which the price changes.

Similarly, the firm can expect to be exposed to a changing demand for their finished product. Recall that the actual demand q_f is a function of the sales price and characteristic parameters b_1 and b_2. We relax the constancy assumption of b_2 and allow this parameter to

vary over time. This can be thought of as a change in demand due to a change in market composition due either to brand switching or a general increase/decrease in the number of market participants. Based on a steadily rising population we will functionally represent this demand as

$$q_f(t) = -b_1 p_f(t) + b_2(t)$$

where

$$b_2(t+1) = b_2(t) + l b_2(t)$$

and the constant l represents the rate at which b_2 changes.

In a dynamic environment we are concerned with the amount of product the firm has at time, t. This amount represents the difference between the quantity produced and the quantity sold which we represent as

$$h_T(t+1) = h_T(t) + q_r(t) - q_f(t)$$

where h_T is the quantity of finished goods available in any period, t. This relationship allows for situations where either the quantity demanded exceeds the quantity supplied or the quantity produced exceeds the quantity demanded. In this manner the model is extended to allow for the consideration of either excess inventory or backlog situations. We assume that backlog will be erased in the following production period and the model

implicitly penalizes the firm in such a situation by incurring the attendant rise in production cost required to fill the backorder units which are sold at the (presumably) lower price contracted for in the prior period. In addition, we penalize the firm an amount m per squared unit of h_T which can be interpreted to represent the inventory carrying cost or the potential loss of customer goodwill due to the firm's inability to satisfy current demand. The dynamic model of the firm can be summarized as:

Demand Relationship

$$q_f(t) = -b_1 p_f(t) + b_2(t) \tag{2-3a}$$

$$b_2(t+1) = b_2(t) + 1b_2(t) \tag{2-3b}$$

Revenue

$$R(t) = p_f(t) q_f(t) \tag{2-4}$$

Cost

$$C_M + C_p(t) = C_M + a_1 + p_r(t)\; q_r(t) + a_2 q_r^{\,2}(t) \tag{2-5a}$$

$$p_r(t+1) = p_r(t) + kp_r(t) \tag{2-5b}$$

Profit

$$P(t) = R(t) - C_M - C_p(t) - mh_T^{\,2}(t) \tag{2-6}$$

Quantity Differential

$$h_T(t+1) = h_T(t) - q_f(t) + q_r(t) \tag{2-7}$$

The firm's problem becomes

Maximize:

$$P(t) = p_f(t)q_f(t) - C_M - a_1 - p_r(t)q_r(t) - a_2 q_r^2(t) - mh_T^2(t) \quad \text{(C)}$$

Subject to:

$$q_f(t) = -b_1 p_f(t) + b_2(t)$$

$$q_r(t) = q_f(t) - h_T(t)$$

$$b_2(t+1) = b_2(t) + lb_2(t)$$

$$p_r(t+1) = p_r(t) + kp_r(t)$$

$$h_T(t+1) = h_T(t) - q_f(t) + q_r(t)$$

where the equation

$$q_r(t) = q_f(t) - h_T(t)$$

represents a constraint on the production decision to recognize current inventory assets or backlog commitments.

This is a more difficult problem to solve than the static model posed for problem B. We are now faced with a situation where the firm recognizes that it must operate in a constantly changing environment and wishes to establish an optimal decision policy over some future planning period. The above formulation allows for future control over any finite number of periods. For exposition purposes we will assume the firm's planning horizon extends five periods into the future. This will provide

enough time to observe dynamic characteristics of the model yet not force the solution to become cumbersome.

The firm's problem can be restated by incorporating the demand relationship into the model as

Maximize:

$$-b_1 p_f^2(t) + b_2(t) p_f(t) - p_r(t) q_r(t) - a_2 q_r^2(t) - m h_T^2(t) - (C_M + a_1) \qquad (C')$$

Subject to:

$$b_2(t+1) = b_2(t) + l b_2(t)$$

$$p_r(t+1) = p_r(t) + k p_r(t)$$

$$h_T(t+1) = h_T(t) + b_1 p_f(t) - b_2(t) + q_r(t)$$

$$q_r(t) = q_f(t) - h_T(t).$$

It should be noted that the model still does not allow us to address the issue of optimal information structure nor does it explicitly address the transfer pricing problem. On the other hand, we can now discuss the dynamic decision-making policy a firm would undertake if we could solve problem C'. In the next section we discuss the theory of optimal control which will provide a means to solve the dynamic problem we have formulated.

Discrete-Time Minimum Principle

The theory of optimal control was developed primarily over the last two decades. This development was two-fold in that it began in this country through the development of dynamic programming and Bellman's principle of optimality (Bellman:1957). Concurrently, a parallel development in the Soviet Union using a different theoretical approach was carried on by Pontryagin and culminated in the minimum principle which is essentially an extension of the calculus of variations approach (Pontryagin, et al.:1962).

Essentially, an optimal control problem consists of:

1. a set of differential or difference equations that represent a system that is to be controlled;

2. a set of constraints on the variables of the system;

3. a set of boundary conditions on the variables; and

4. a cost functional, or performance index, which is to be maximized/minimized.

The application of this theory to our problem is straightforward. The system is represented by a model of the firm, a set of difference equations. Our model incorporates explicit constraints on the variables of the

system and requires boundary conditions on the initial values of the variables. Finally, the cost function is represented by the decision maker's goals, objectives or utility. The objective of this section is to discuss a solution for the discrete-time optimal control problem. The state-space approach will be used extensively in this and remaining sections of the dissertation. The reader who is not familiar with this approach should read Appendix A before proceeding and is also referred to Ogata (1967) for an extensive treatment.

Since most of the early applications of control theory to engineering problems involved continuous time systems, the theoretical foundations for optimal control developed most extensively in the continuous-time form. The minimum principle of Pontryagin which provides a set of necessary conditions for the solution of the general continuous-time optimal control problem has found wide acceptance and application to engineering problems. One purpose of this section of the dissertation is to discuss a minimum principle for discrete-time optimal control problems that will be general enough to allow application to the problems that will interest us.

Pearson and Sridhar (1966) and Rosen (1967) have shown that the minimum principle could be approached from the point of view of Kuhn-Tucker theory and that a dynamic

optimal control problem could be expressed and treated as a larger static convex programming problem. We will use their approach in deriving a minimum principle for discrete-time problems. The basic problem in convex programming is that of minimizing the scalar function $J(y)$ subject to the constraints $F_1(y) = 0$ and $F_2(y) \geq 0$ where y is an s-vector, $F_1(y)$ is an n_1 dimensional vector valued function and $F_2(y)$ is a vector valued function of dimension n_2. In addition, the assumptions are made that $J(y)$, $F_1(y)$ and $F_2(y)$ are all differentiable in their arguments and that the constraint functions are convex in y.

The results of Kuhn-Tucker theory that are of interest are two theorems that state conditions for the solution of the convex programming problem. Define the Lagrangian as:

$$L(y,p,\mu) = J(y) + p^T F_1(y) - \mu^T F_2(y)$$

where p and μ are n_1 and n_2 vectors of Lagrange multipliers. Assume that y^* is an admissible value which satisfies the constraints and minimizes $J(y)$ and define the following vectors:

$$L_y^* = \partial L/\partial y_i|y^*,p^*,\mu^*; \; i = 1,2,\ldots,s$$

$$L_p^* = \partial L/\partial p_i|y^*; \; i = 1,2,\ldots,n_1$$

$$L_\mu^* = \partial L/\partial \mu_i|y^*; \; i = 1,2,\ldots,n_2$$

L_y^* is the gradient vector of the Lagrangian with respect to y (i.e., each of the s components of y) evaluated at $y = y^*$, $p = p^*$, and $\mu = \mu^*$; i.e., at the value that minimizes $J(y)$ while satisfying the constraints $F_1(y) = 0$ and $F_2(y) \geq 0$ and the corresponding values of p and μ. L_p^* and L_μ^* are the gradient vectors with respect to p and μ, evaluated at $y = y^*$.

For convenience, define the two matrices:

$$F_{1y}^* = \partial F_{1i}/\partial y_j|y^*;\ i = 1,2,\ldots,n_1;\ j = 1,2,\ldots,s$$

$$F_{2y}^* = \partial F_{2i}/\partial y_j|y^*;\ i = n_1+1,\ldots,n_1+n_2;\ j = 1,2,\ldots,s.$$

Thus we have

$$L_y^* = \partial J/\partial y|y^* + (F_{1y}^*)^T p^* - (F_{2y}^*)^T \mu^*$$

$$L_p^* = F_1(y^*)$$

$$L_\mu^* = -F_2(y^*).$$

The two Kuhn-Tucker theorems of interest are:

<u>Theorem I</u>:

If y^* minimizes $J(y)$ subject to $F_1(y) = 0$ and $F_2(y) \geq 0$, then it is necessary that there exist some p^* and μ^*, so that the following are satisfied:

$$L_y^* = 0$$

$$L_p^* = 0$$

$$L_\mu^* \leq 0$$

$$(L_\mu^*)^T \mu^* = 0$$

$$\mu^* \geq 0.$$

Theorem II:

If y^* minimizes $J(y)$ subject to $F_1(y) = 0$ and $F_2(y) \geq 0$, it is sufficient that Theorem I holds and

$$L(y,p^*,\mu^*) \geq L(y^*,p^*,\mu^*) + (L_y^*)^T(y-y^*).$$

(Note that Theorem II is merely a convexity condition on L.)

Theorem I gives us a set of necessary conditions for the solution to the optimization problem that may admit several extremal solutions; however, Theorem II states that if the optimization problem is such that the Lagrangian has a unique minimum with respect to y, then there is only one extremal solution and Theorem I gives sufficient conditions for an optimum.

Now we outline the optimal control problem as composed of the system

$$x(t+1) - x(t) = f(x(t),u(t),t);\ t = 0,1,\ldots,N$$

where $x(t)$ is now an n-vector and $u(t)$ is an r-vector. The system is subject to the initial conditions

$$x(t=0) = x(0)$$

and the final condition

$$t = N.$$

We want to minimize the cost functional

$$J = K(x(N)) + \sum_{t=0}^{N-1} L(x(t),u(t),t)$$

and we require that the sequence $\{x(t),u(t)\}$ belongs to the constraint set respresented as

$$\rho(x(t),u(t),t) \geq 0$$

where ρ is a vector-valued function of dimension m.

To apply the Kuhn-Tucker theorems, the optimal control problem must be restated as a convex programming problem. To do this, define the $(n+r)N = sN$ vector y as

$$y = [x(1),\ldots,x(N),u(0),\ldots,u(N-1)]^T.$$

Next define the nN vector $F_1(y)$ as

$$F_1(y) = \begin{bmatrix} f(x(0),u(0),0) - x(1) + x(0) \\ f(x(1),u(1),1) - x(2) + x(1) \\ \vdots \\ f(x(N-1),(N-1),N-1) - x(N) + x(N-1) \end{bmatrix}$$

and the mN vector $F_2(y)$ as

$$F_2(y) = \begin{bmatrix} \rho(x(0),u(0),0) \\ \vdots \\ \rho(x(N-1),u(N-1),N-1) \end{bmatrix}$$

Thus, the optimal control problem is equivalent to minimizing:

$$J(y) = K(x(N)) + \sum_{t=0}^{N-1} L(x(t),u(t),t)$$

subject to:

$$F_1(y) = 0$$

$$F_2(y) \geq 0.$$

We can now apply the Kuhn-Tucker theorems stated earlier where we define the Lagrangian as

$$L(y,p,\mu) = J(y) + p^T F_1(y) - \mu^T F_2(y)$$

However, p is now an nN-vector and u is an mN-vector of Lagrange multipliers. Application of the

Kuhn-Tucker conditions results in the following discrete-time minimum principle.[5]

Discrete Minimum Principle

Let $\{x^*(t)\}$ be the trajectory of the dynamic system of interest corresponding to the control sequence $\{u^*(t)\}$ with $x^*(t=0) = x(0)$ and $\{x^*(t), u^*(t)\}$ constrained to the set of $\rho(x(t), u(t), t) \geq 0$. Then if $\{u^*(t)\}$ minimizes the cost functional $J = K(x(N)) + \sum_{t=0}^{N-1} L(x(t), u(t), t)$, it is necessary that there exists a sequence of n vectors $\{p^*(t);\ t = 0,1,\ldots,N\}$ called the co-states, and a sequence of m vectors $\{\mu^*(t);\ t = 0,1,\ldots,N\}$ called the co-constraint vectors such that:

1. The scalar function

$$\begin{aligned} H(x^*(t), p^*(t+1), u(t), \mu^*(t+1)) &= L(x^*(t), u(t), t) \\ &+ (p^*(t+1))^T f(x^*(t), u(t), t) \\ &- (\mu^*(t+1))^T \rho(x^*(t), u(t), t) \end{aligned} \tag{2-8}$$

called the Hamiltonian is minimized as a function of $u(t)$ at $u(t) = u^*(t)$ for all $t = 0,1,\ldots,N-1$.

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[5]The interested reader is referred to Pindyck (1973) for the algebraic details.

2. The dynamics of $x^*(t)$, $p^*(t)$, and $\mu^*(t)$ are determined by the equations

$$x^*(t+1) - x^*(t) = \partial H/\partial p(t+1)|_* = f(x^*(t),u^*(t),t) \tag{2-9}$$

$$p^*(t+1) - p^*(t) = -\partial H/\partial x(t)|_* \tag{2-10}$$

$$\rho(x^*(t),u^*(t),t) = -\partial H/\partial \mu(t+1)|_* \geq 0 \tag{2-11}$$

$$\mu^*(t) \geq 0 \tag{2-12}$$

$$\rho^T(x^*(t),u^*(t),t)\mu^*(t+1) = 0 \tag{2-13}$$

$$p^*(N) = \partial K(x^*(N))/\partial x. \tag{2-14}$$

<u>Model Revisited</u>

For case of reference we restate the firm's problem as defined by equations C':

Maximize:

$$-b_1 p_f^{\,2}(t) + b_2(t)p_f(t) - p_r(t)q_r(t) - a_2 q_r^{\,2}(t) - mh_T^{\,2}(t) - (C_M+a_1) \tag{C'}$$

Subject to:

$$b_2(t+1) = b_2(t) + lb_2(t)$$

$$p_r(t+1) = p_r(t) + kp_r(t)$$

$$h_T(t+1) = h_T(t) - b_2(t) + b_1 p_f(t) + q_r(t)$$

$$q_r(t) = q_f(t) - h_T(t).$$

Recall that this model assumes the availability of perfect information on which the centralized form will base its production decision and product line pricing policy. For mathematical convenience we will restate problem C' in state-space representation which will enable us to formulate the firm's problem as an optimal control problem. Let us define the three-dimensional state vector

$$x(t) = \begin{bmatrix} b_2(t) \\ p_r(t) \\ h_T(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \text{demand parameter} \\ \text{raw material price} \\ \text{inventory/backlog} \end{bmatrix} .$$

Similarly, we define the two-dimensional decision/policy/control vector as

$$u(t) = \begin{bmatrix} q_r(t) \\ p_f(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \text{production decision} \\ \text{pricing decision} \end{bmatrix} .$$

We can now generate the state-space representation of the model of the firm as

$$x(t+1) - x(t) = f(x(t),u(t),t)$$

where the (3x1) vector valued function f is defined as

$$f(x(t),u(t),t) = \begin{bmatrix} 1x_1(t) \\ kx_2(t) \\ -x_1(t) + b_1u_2(t) + u_1(t) \end{bmatrix} .$$

We note that the model of the firm can be expressed in linear form as

$$x(t+1) - x(t) = Ax(t) + Bu(t) \qquad (2\text{-}15)$$

where the (3x3) matrix A and the (3x2) matrix B are defined as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ -1 & 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & b_1 \end{bmatrix} .$$

The dynamic model of the firm we have developed is completely described by the vector-matrix difference equation (2-15). The cost function was defined earlier as

$$J = K(x(N)) + \sum_{t=0}^{N-1} L(x(t),u(t),t).$$

We can convert the firm's objective of profit maximization to the analogous cost minimization problem by defining

$$J = -(1/2)K(x(N)) - (1/2)\sum_{t=0}^{N-1} P(t)$$

where P(t) is defined by equation (2-6) and K defines the final state of the model which yields

$$J = (1/2)\sum_{t=0}^{N-1} (C_M + C_P(t) + mh_T^{\,2}(t) - R(t))$$
$$+ (1/2)mh_T^{\,2}(N)$$

which can be written explicitly as

$$J = (1/2)\sum_{t=0}^{N-1} (b_1 p_f^{\,2}(t) - b_2(t)p_f(t) + p_r(t)q_r(t)$$
$$+ a_2 q_r^{\,2}(t) + mh_T^{\,2}(t) + (C_M + a_1))$$
$$+ (1/2)\ mh_T^{\,2}(N).$$

Converting to state-space representation yields

$$J = (1/2)\sum_{t=0}^{N-1} (b_1 u_2^{\,2}(t) - x_1(t)u_2(t) + x_2(t)u_1(t)$$
$$+ a_2 u_1^{\,2}(t) + mx_3^{\,2}(t) + (C_M + a_1))$$
$$+ (1/2)mx_3^{\,2}(N)$$

which can be written in matrix form as

$$J = (1/2) \sum_{t=0}^{N-1} (x^T(t)Qx(t) + u^T(t)Ru(t) + x^T(t)Su(t) + C) + (1/2)x^T(N)Qx(N)$$

where x(t) and u(t) are the state and control vectors pre-variously defined and C = constant scalar $(C_M + a_1)$ with

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$R = \begin{bmatrix} a_2 & 0 \\ 0 & b_1 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

We now restate the firm's problem as an optimal control problem (problem D) in which we want to select a control sequence $\{u^*(t)\}$ such that the function

$$J = (1/2) \sum_{t=0}^{N-1} (x^T(t)Qx(t) + u^T(t)Ru(t) + x^T(t)Su(t) + C) + (1/2)x^T(N)Qx(N) \qquad \text{(D)}$$

is minimized and the system

$$x(t+1) - x(t) = Ax(t) + Bu(t)$$

is subject to the initial condition

$$x(t=0) = x(0)$$

and the final condition

$$t = N$$

and the quantity constraint

$$q_r(t) = q_f(t) - h_T(t).$$

We can apply the minimum principle we developed in the previous section to determine an optimal planning policy for the multi-period model.

Dynamic Optimization

Application of the discrete-time minimum principle to our multi-period model yields the following optimal planning policy:[6]

$$u^*(t) = J\, x^*(t) \tag{2-16a}$$

where

$$J = (1/2)(DR^{-1}D^T)^{-1}\, R^{-1}D^TDR^{-1}S^T - (1/2)R^{-1}S^T - (DR^{-1}D^T)^{-1}R^{-1}D^TC. \tag{2-16b}$$

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[6] See Appendix B for derivation.

This relationship shows that the optimal planning decisions are a linear function of the current state of the system. For our model J becomes

$$J = 1/(2(1+a_2b_1)) \begin{bmatrix} 1 & -b_1 & -2 \\ (1+2a_2b_1)/b_1 & 1 & -2a_2 \end{bmatrix}$$

thus the optimal planning policy is determined by

$$u^*(t) = 1/(2(1+a_2b_1)) \begin{bmatrix} 1 & -b_1 & -2 \\ (1+2a_2b_1)b_1 & 1 & -2a_2 \end{bmatrix} \begin{bmatrix} x^*(t) \end{bmatrix} \qquad (2\text{-}17)$$

If we assume that the initial inventory/backlog is zero and ignore the dynamics of our model, we note that the optimal planning policy becomes

$$u^* = 1/(2(1+a_2b_1)) \begin{bmatrix} 1 & -b_1 & -2 \\ (1+2a_2b_1)/b_1 & 1 & -2a_2 \end{bmatrix} \begin{bmatrix} b_2 \\ p_r \\ 0 \end{bmatrix}$$

$$u^* = 1/(2(1+a_2b_1)) \begin{bmatrix} b_2 - b_1p_r \\ (b_2(1+2a_2b_1))/b_1 + p_r \end{bmatrix} = \begin{bmatrix} q_r \\ p_f \end{bmatrix} . \qquad (2\text{-}18)$$

Expanding equation (2-18) we observe that the optimal production decision is

$$q_r^* = \frac{1}{2(1+a_2b_1)} (b_2-b_1p_r) \qquad (2\text{-}18a)$$

This result is identical to the classical result obtained in equation (2-1). We note that the optimal production decision is linearly related to the demand parameter b_2 in a positive manner such that an increase in b_2 (which implies a change in demand schedule) results in an increase in the production decision. This result is intuitively appealing since we would expect an increase in the quantity demanded to result in an increase in the production decision. The second demand parameter b_1 (recall that we have posited a linear demand function of the form $q_f = -b_1 p_f + b_2$) appears in both the numerator and denomerator of the optimal production decision. We note that a perfectly competitive external market requires that $b_1 = 0$. In this limiting case, the optimal production decision becomes $b_2/2$ which represents the upper limit for optimal production, since an increase in b_1 reduces the magnitude of the numerator and increases the magnitude of the denominator. Thus, as b_1 increases (which implies that customer demand is becoming more sensitive to pricing considerations) the production decision becomes more conservative. This result captures a conservative aspect of the model that adjusts the production decision downward (which results in a hedge against the risk of losses due to overproduction) as the demand for the product becomes more volatile.

Further analysis of the optimal production decision reveals the expected results with respect to the production cost components. Increases in the raw material cost, p_r, or the internal variable cost parameter, a_2, both result in reductions in production.

Further expansion of equation (2-18) reveals the optimal pricing decision as

$$p_f^* = \frac{1}{b_1}(b_2 - q_r^*). \tag{2-18b}$$

This result is identical to the classical result of equation (2-2). We observe that demand parameters b_2 and b_1 appear in the numerator and denominator, respectively. Thus we observe the same affect on the optimal pricing decision as was seen for the optimal production decision. That is, an increase in the demand schedule results in an increase in the product price whereas a consumer market that becomes more volatile with respect to pricing considerations, results in a reduction of the price of the product. These results are intuitively appealing in that we tend to observe large firms acting in the manner discussed here. We also note that the dynamic, multi-period results can be readily reduced to the classical economic results that the firm should price its product such that marginal revenue equals marginal cost. The multi-period analysis under conditions of certainty can be iteratively

solved as a static problem in each period using the results of equation (2-18). In fact, if our purpose was to design a planning model under conditions of certainty, we could have done so without the use of an optimal control theory approach. However, we are interested in developing a control technique which will result in encouraging decision makers to act in such a manner as to implement decisions that are in conformance with some overall corporate plan. For illustrative purposes, assume that the firm has decided on the following five-year corporate plan:

	Year 1	Year 2	Year 3	Year 4	Year 5
Production Level	1448	1541	1639	1743	1853
Product Price	\$35520	\$37090	\$38733	\$40452	\$42250

The above plan was actually generated using the planning model developed in this chapter, conditioned on a current raw material price of \$20750, demand parameters $b_1 = 0.1$ and $b_2(current) = 5000$, growth rates for 1 and k of 5 percent and 3 percent respectively and a penalty, m = \$10000.[7] The actual planning process that management uses to arrive at the desired targets is not crucial to this discussion. The essential problem we are interested in is, "Given the planning objectives, can we derive a

[7]See Appendix C for computation of plan.

control technique which will result in decentralized decisions that result in conformance to the overall corporate plan?"

In this chapter we have developed a model for centralized decision making. In the next chapter we extend the model to incorporate decentralized decision making and develop a control technique that will encourage decentralized decision makers to implement actions that are not only in their best interests, but also in the best interest of the firm overall.

CHAPTER III

DECENTRALIZED CONTROL MODEL

Introduction

In Chapter II we developed an optimal planning model for a firm that operates in a deterministic centralized decision-making environment. In this chapter we will develop a control technique that will encourage decision makers to achieve the planning objectives which the firm has established. This control technique will then be applied to a decentralized extension of the model developed in the last chapter. The control technique will be evaluated with respect to its ability to encourage decentralized decision makers to act in a manner that is in the best interest of the firm as a whole.

Control Technique

In this section we discuss a control technique that has been used in the design of physical systems as a result of the application of optimal control theory to a linear dynamic system with a quadratic performance index used as the instrument to measure the desired performance of the system. The control technique is also applicable to stochastic systems (as we shall discuss later) and is

commonly referred to as the Linear-Quadratic-Gaussian (LQG) problem.[8] A discrete version of the control philosophy inherent in the LQG problem with emphasis on economic system analysis was presented in 1972 (Athans:1972). Since that time economists have applied the control approach to numerous problems concerned with the stabilization and control of economic systems.[9] The basic control mechanism is a performance index intended to encourage conformance with some predetermined nominal plan.[10] Kornai and Simonovits (1977) have addressed this control philosophy by defining a real sphere and a control sphere where the real sphere consists of the dynamic model and its desired objectives and the control sphere involves a penalty function used to measure the performance of the system. The essential idea of the control technique is to define a penalty function, quadratic in form, that will punish deviations from a desired plan. The quadratic form results in small penalties for small deviations and

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[8] For a comprehensive survey of engineering applications, see the Special Issue of IEEE Transactions on Automatic Control (Athans:1971).

[9] For a survey of economic applications, see Kendrick (1976).

[10] As a historical point, it is interesting to note that the basic concept of LQG control was initially introduced in an industrial setting by Holt, et al. in 1960.

increasingly higher penalties for more significant deviations. In the next section we will develop a control model for decision-making using the LQG approach (with the exception that our model does not yet explicitly incorporate uncertainty considerations).

Control Model

We expressed the dynamic model of the centralized firm as:

$$b_2(t+1) - b_2(t) = 1\ b_2(t)$$

$$p_r(t+1) - p_r(t) = k\ p_r(t)$$

$$h_T(t+1) - h_T(t) = -b_2(t) + b_1 p_f(t) + q_r(t)$$

subject to the constraint

$$q_r(t) = q_f(t) - h_T(t)$$

and developed an optimal control problem in which we wanted to select a decision policy, $\{u^*(t)\}$, such that the function

$$J = (1/2) \sum_{t=0}^{N-1} (x'(t)Qx(t) + u'(t)\ Ru(t)$$

$$+ x'(t)Su(t) + C) + (1/2)x'(N)Qx(N)$$

was minimized subject to the dynamic model of the firm and the static constraint.

Application of the discrete minimum principle resulted in an optimal policy for the firm (Eqn. 2-18). This policy was used to generate sample corporate targets for production levels and product pricing. We extend our model to introduce the concept of separation of ownership and managerial decision making by defining a predetermined plan specified by the owner of the firm. We assume that the (centralized) decision maker attempts to achieve the owner's objectives due to an agreed upon incentive arrangement based on penalizing the decision maker for deviations from the owner's plan. This concept can be formulated as a control problem by defining a penalty function of the form:

$$
\begin{aligned}
J_c = (1/2) \sum_{t=0}^{N-1} & ((x(t)-\bar{x}(t))' Q_c (x(t)-\bar{x}(t)) \\
& + (u(t)-u(t))' R_c (u(t)-\bar{u}(t))) \\
& + 1/2 (x(N)-\bar{x}(N))' Q_c (x(N)-\bar{x}(N))
\end{aligned}
\tag{3-1}
$$

where the planning objectives, $\bar{x}(t)$ and $\bar{u}(t)$, are incorporated into the penalty function and the matrices, Q_c and R_c, are used to "weight" the relative importance of both state deviations and control/decision deviations (note that these are not the same Q and R matrices defined earlier). This penalty function must be minimized subject to the actual dynamic model of the firm developed

earlier; i.e.,

$$x(t+1) - x(t) = A\,x(t) + B\,u(t) \tag{E}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ p_r \\ h_T \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} q_r \\ p_f \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & b_1 \end{bmatrix}.$$

The solution to this problem can be obtained using the discrete minimum principle in the same manner as was done in Appendix B. The resulting optimal control policy for the decision maker to follow is

$$u^*(t) = -(R_c+B'K(t+1)B)^{-1}B'K(t+1)((I+A)x^*(t) + B\bar{u}(t)) + \bar{u}(t) \tag{3-2}$$

$$K(t) = Q_c + (I+A)'(K(t+1)-K(t+1) B(R_c+B'K(t+1)^{-1}B'K(t+1))(I+A) \tag{3-3}$$

$$K(N) = Q_c. \tag{3-4}$$

The selection of the weighting matrices in the quadratic criterion is not a simple matter. These matrices provide a mechanism to operationalize the technique by which the owner provides an incentive arrangement based on his preferences. It should be noted that in this control model, the goals $\bar{x}(t)$ and $\bar{u}(t)$ need not be dependent upon each other, nor do they have to be generated using the dynamic system (E). Furthermore, the static constraint was not considered in the development of the control model. We will have more to say concerning the incentive arrangement (weighting matrices) when we extend our model to a decentralized environment. In most practical applications we will select Q_c and R_c to be diagonal. In this manner specific components of the state deviations and control deviations can be weighted individually and their impact can be assessed quantitatively.

As a check on our model we use the plan developed in Chapter II and (somewhat arbitrarily) set

$$Q_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This incentive arrangement weights the deviations from quantity differential, price and production quantity equally and assigns a zero weight to the demand parameter and the input product price. Since the firm has no control over these last two variables, we would not expect to see any incentive arrangement with respect to the raw material price or the demand parameter, $b_2(t)$.

Using Eqns. (3-2) through (3-4) we find that

$$u^*(t) = \bar{u}(t). \tag{3-5}$$

Thus the control model we developed results in actions by the decision maker which follow the owner's plan exactly. Although this simple example verifies the correctness of the control model it does not approximate reality in that the owner seldom specifies exactly the quantities of production and the sales price (if this could be done the role of the decision maker would no longer be required). Instead, we observe targets in the form of cost performance (budgets) and revenue performance (sales quotas). Thus the owner can establish targets for cost and revenue without the detailed knowledge of internal performance parameters. The decision maker would then internalize these goals by defining explicit objectives internal to the firm. For example, if the owner establishes a revenue target of r(t), the decision maker,

having full knowledge of the firm's operations, will internalize the goal of $p_f(t)q_f(t) = r(t)$ and thereby establish an internal target $p_f(t)$. In this way the control model can be generalized to allow the owner to establish incentive mechanisms for targets which he desires the decision maker to achieve and the decision maker (through the model dynamics and knowledge of the interrelationships internal to the firm) will internalize those targets by defining explicit internal objectives.

Decentralized Model of the Firm

Figure 3.1 extends the earlier framework (Figure 2.1) to address a decentralized organization.

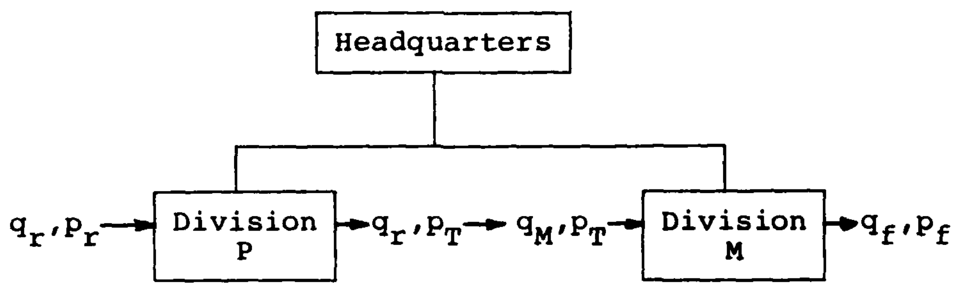

Fig. 3.1. Decentralized Model of the Firm

This framework incorporates two significant extensions of the earlier model. First, an amount q_M is transferred from Division P to Division M for which Division M pays Division P an amount p_T, the transfer price. Second, Division P has the decentralized authority to make the

production decision, q_r, and the transfer pricing decision, p_T. The manager of Division M, in turn, has the decentralized authority to decide on the amount of the transferred good, q_M, he will purchase and the external product pricing decision, p_f. Recent literature on transfer pricing (see Chapter I) has essentially advocated some form of negotiation scheme involving an iterative process which may, or may not, involve corporate headquarters. However, these analyses have primarily dealt with a deterministic, static environment. The dynamic, multi-period analysis we are investigating will tend to minimize any "gaming" strategies during a negotiation process since there is ample time for future "settling up." Since any iterative process is, by definition, time-consuming we wish to investigate the possibility of establishing some incentive scheme that will encourage the decentralized decision makers to arrive at decisions that are in the best interest of the firm overall without requiring a time-consuming process. A significant premise of this dissertation is that the decentralized decision makers can be provided incentives such that it is in their best interest to work as a team to achieve the overall corporate goals. The team approach was introducted by Marschak and Radner (1972). The team concept essentially suggests that functional behavior in a large decentralized

organization dominates any dysfunctional behavior, such that inherent game-theoretic organizational traits can be disregarded for analytic purposes.

Our decentralized model also assumes that the intermediate market, i.e., the internal transfer of goods, is restricted to one buyer (Division M) and one seller (Division P) and can thus be considered as a bilateral monopoly. The familiar Edgeworth Box analysis of this situation dictates that a mutually satisfactory solution will lie along a contract curve reached as a result of mutual benefit to both the buyer and the seller. Although this form of equilibrium (referred to as Pareto optimality) can be achieved in principle, it is not apparent that a Pareto optimal solution would be in the best interests of the firm as a whole. It is well known that the classical economic solution for this situation is indeterminate without the addition of negotiation or an incentive scheme. Dopuch and Drake (1964) have shown that other market situations readily lead to optimal solutions which dictate the use of the prevailing market price if the intermediate market is perfectly competitive or the use of Hirshleifer's procedure (Hirschleifer:1956) if the competition is imperfect.

Our decentralized model generates a dynamic system in which several decision maker's actions will jointly

affect the dynamic behavior of the system. The decision makers will base their actions on partial and (in the sequel) imperfect information on the various states of the dynamic system on each other's actions. In a decentralized environment we envision the production manager to have access to current information that is not available to the marketing manager, internal production cost information, for example. Similarly, we would expect that the marketing manager has access to current market information not available to the production manager. Thus, the information necessary to make "optimal" decisions is decentralized and is not available in any one place. This situation represents a radical departure from Walrasian systems in which all the necessary information is assumed to be available to the auctioneer or to the central agency (headquarters). Since all of the needed information is not available in any one place, the control of a decentralized organization is more difficult than for a centralized organization. A decentralized information pattern implies certain structural restrictions on control policies. This lack of centralized information requires a degree of cooperation among decision makers so that their actions can be coordinated to work together to control the decentralized dynamic system. Thus the problem of controlling a decentralized organization involves team

decision making which is a special case of the theory of teams (Aoki:1976).

The dynamic model of the decentralized firm can be expressed as

$$
\begin{aligned}
p_r(t+1) &= (1+k)p_r(t) \\
b_2(t+1) &= (1+l)b_2(t) \qquad\qquad (F) \\
h_{TM}(t+1) &= h_{TM}(t) + q_M(t) - q_f(t) \\
&= h_{TM}(t) + q_M(t) + b_1 p_f(t) - b_2(t) \\
h_{TP}(t+1) &= h_{TP}(t) + q_r(t) - q_M(t)
\end{aligned}
$$

where $h_{TM}(t)$ and $h_{TP}(t)$ represent the difference between the quantity of goods "produced" and the quantity sold for the marketing and production divisions, respectively. The dynamics of the decentralized model become more complex than that of a centralized firm due to the addition of an additional dynamic equation (due to quantity differentials). Furthermore, decisions are now made by different decision makers; i.e., Division P has control of $q_r(t)$ and $p_T(t)$ whereas Division M has control of decisions $q_M(t)$ and $p_f(t)$.

As discussed earlier, this model does not have a determinate optimal economic solution. To determine the explicit impact of transfer pricing policy on the decentralized decision maker's actions, we modify the model to

enable us to evaluate the implications of transfer pricing policy by describing the transfer price, p_T, as an additional state variable as opposed to a decentralized decision variable. This modification results in the following decentralized decision-making model:

$$
\begin{aligned}
p_r(t+1) - p_r(t) &= kp_r(t) \\
p_T(t+1) - p_T(t) &= \theta p_T(t) \\
b_2(t+1) - b_2(t) &= lb_2(t) \qquad\qquad (G) \\
h_{TM}(t+1) - h_{TM}(t) &= q_M(t) + b_1 p_f(t) - b_2(t) \\
h_{TP}(t+1) - h_{TP}(t) &= q_1(t) - q_M(t).
\end{aligned}
$$

The interpretation of this model is that the transfer price, $p_T(t)$, is no longer controllable by a decentralized decision maker but has been established by the corporate headquarters. In this manner, the effect of exogenous transfer price changes can be evaluated with respect to their impact on decentralized decision making. In addition, we have not burdened this model, (G), with any of the many possible constraints that could enter into an internal action because we wish to minimize any informational requirements inherent in the model and allow as much flexibility as possible for decentralized decision making. The decentralized decision-making model can be represented as

$$x(t+1) - x(t) = A\,x(t) + \sum_{i=1}^{2} B_i u_i \tag{H}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} p_r \\ p_T \\ b_2 \\ h_{TM} \\ h_{TP} \end{bmatrix} \;;\; u_1 = \begin{bmatrix} q_r \end{bmatrix} \;;\; u_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} q_M \\ p_f \end{bmatrix}$$

$$A = \begin{bmatrix} k & 0 & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \;;\; B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & b_1 \\ -1 & 0 \end{bmatrix}$$

In this model the decentralized decision making is represented by $u_1(t)$ - the production division decisions, and $u_2(t)$ - the marketing division decisions.

Information Structure

The decentralized aspects of information availability will be operationalized by defining an information structure, $I_j(t)$, where $j = 1,2$ for the representative decision makers and

$$I_j(t) = \{y_j(t),\ Y(t-1),\ u(t-1)\}$$

where $y_j(t)$ represents the current information available to decision maker j and $Y(t-1)$, $u(t-1)$ represent the past history of the organization; that is $Y(t-1) = \{y(1), y(2), \ldots, y(t-1)\}$ and $u(t-1) = \{u(1), u(2), \ldots, u(t-1)\}$. This information structure is referred to as one-step delay sharing information structure in the information and control theory literature (Witsenhausen:1971). Thus each division has access to some subset of the current state information and is aware of prior states and decisions. This situation closely models the physical environment where a decentralized divisional decision maker would be aware of past actions taken by decision makers he interacts with and also has access to some current information concerning the state of the organization. This current information $y_j(t)$ can be expressed as

$$y_j(t) = H_j(t)\ x(t). \qquad \text{(I)}$$

The information matrix, $H_j(t)$, is used to explicitly recognize the extent to which information is decentralized. In the prior centralized model we did not recognize an observation state, $y(t)$, due to the implicit assumption that all information needed was available. A similar situation, i.e., perfect information completely available, would result if $H_j(t)$ is set equal to the identity matrix.

This results in a situation where the information structure is completely centralized.

Decentralized Control

Since classical stochastic control is restricted to a single decision maker, the need for a borader theory to address decentralized control problems is apparent. The current state of decentralized control theory is in its infancy (Athans:1978).[11] A brief summary of the evolution of decentralized control theory was discussed by Basar (1978) as:

> The first decentralized result . . . has been obtained by Radner (1962) who has shown among other things that a static LQG team problem admits a unique team-optimal solution linear in the observation of each decision maker. This result, however, is to be interpreted with caution when the information structure is dynamic and nonclassical. The famous counterexample of Witsenhausen (1968) is indicative of this fact, that the team-optimal solution of a dynamic 2-member team problem with 2-step delay information will in general not be linear. Ho and Chu (1972), Chu (1972), and Chu and Ho (1971) have studied nonclassical but nested information structures and have applied within that context Radner's above cited result to dynamic LQG problems. The first systematic formulation of decentralized stochastic team problems within a general framework has been given by Witsenhausen (1971) where he has made several important assertions. One of these assertions was team-optimality of linear solutions in the optimization of dynamic LQG team problems under the one-step delay information sharing pattern. This assertion was then considered almost independently by Kurtaran and Sivan (1974), Sandell and Athans (1974), and Yoshikawa (1975) where the authors adopted a dynamic programming

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[11] See this issue for a review of the current state of decentralized control theory.

approach to derive a set of relations for the linear solution of a 2-member LQG team problem to satisfy.

We noted that the decentralized decision-making model we developed involved a one-step delayed information pattern. The information structure is dynamic in the sense that current decisions are influenced by decisions made by other decision makers. If an information structure depends only on the state observations then it is referred to as static in so much as the decisions of one member are not affected by the decisions of another decision maker. Yoshikawa (1975) has shown that the one-step delay information sharing structure constitutes a dynamic problem that can be decomposed into N static team problems by applying the technique of dynamic programming.

At this point in the development of our model for decentralized decision making, a cursory review of its evolution reveals that we now have a decentralized model for decision making under conditions of certainty. However, the concept of decentralization implies that uncertainty exists with repsect to the information availability since a world of certainty allows centralized decision making. In fact, further analysis of the model, as it now exists, would reveal that it trivially reduces to a situation with a fully centralized information structure.

Thus, we must extend the model to address the problem of d:centralized decision making under conditions of uncertainty to facilitate any further meaningful analysis.

CHAPTER IV

DECENTRALIZED DECISION MAKING UNDER CONDITIONS OF UNCERTAINTY

Introduction

In this chapter we extend our control model to allow for the uncertainty that exists in a decentralized decision-making environment. We then derive an optimal decentralized control policy and apply the results to our specific model to provide a mechanism to discuss information availability and evaluate transfer pricing policy.

Model Uncertainty

Our decentralized model of the firm is a mathematical model of a physical process. The model is an approximation which neglects second-order effects. However, if the model itself was exact structurally, the values of the parameters used in the model would be estimates and may be slightly different from their true values. This uncertainty can be considered in our model by explicitly considering the stochastic nature of the firm. We incorporate uncertainty into our model as a zero-mean disturbance with a known variance. We

represent model uncertainty by the stochastic random variable, $w(t)$, where

$$E\{w(t)\} = 0$$

$$E\{w(t)w'(t)\} = W(t).$$

Thus $w(t)$ is an n-dimensional vector used to incorporate uncertainty into our model. For example, when we assumed the following structural relationship,

$$p_r(t+1) = (1+k)p_r(t), \tag{4-1}$$

we did not admit the presence of any system uncertainty. In actuality, we do not have the ability to discern equation (4-1) precisely. We would be more accurate in writing a relationship of the form

$$p_r(t+1) = (1+k)p_r(t) + w_r(t). \tag{4-2}$$

where $w_r(t)$ is defined above. In this manner we allow for model uncertainties that may exist in the structural equations. The stochastic model of the decentralized firm becomes (from eqn. (H)) and the inclusion of model uncertainty

$$x(t+1) = (I+A)x(t) + \sum_{i=1}^{2} B_i u_i(t) + w(t) \tag{4-3}$$

where $w(t)$ represents the uncertainty associated with the dynamic changes of the state of the firm.

In addition to the model uncertainty, we also recognize the lack of precision inherent in the available current information. We realize that data are frequently used which may be imperfect due to cost or timing considerations. The decision maker must frequently base his decisions not only on incomplete information (decentralized concept) but also on imperfect information that is currently available. This uncertainty can be incorporated into the observation portion of our model in a manner similar to the acknowledgement of system uncertainty. We define the random variable, $v(t)$, where $E\{v(t)\} = 0$, $E\{v(t)v'(t)\} = V(t)$. Then equation (I) can be written as

$$y_j(t) = H_j(t)x(t) + v_j(t). \tag{4-4}$$

The complete stochastic model of the decentralized firm can be represented by equations (4-3) and (4-4). The state-space representation of the model of the firm allows us to explicitly recognize uncertainty in a dynamic environment not only with respect to the structural model but also with respect to the information currently available to the decentralized decision makers.

Decentralized Team Performance Criteria

We stated earlier that a major premise of this study is that an organization can be considered as a team and that incentive arrangements can be made to induce team behavior. In an organization consisting of many members with different information and decision possibilities, it is possible that some organizational objectives may not be consistent with the individual member's objectives. The theory of teams analyzes organizational decision making where different members' decisions may depend on different information, but a common goal exists. Therefore, in standard team problems, there is no incentive problem since there is no conflict of interest. However, Groves (1973) has shown that there exists a system of compensation rules, or incentives, that will induce members of an organization to behave as a team. Groves notes that the head of an organization has some latitude in selecting the rules for compensating his managers and it is desirable for him to select rules, if they exist, that will induce his managers to behave as if they were members of a team. He terms any set of compensation rules an incentive structure and views the role of the organization's head as finding an optimal incentive structure that will induce the managers to behave as if they formed a team. Groves' results apply to what he terms a

conglomerate organization which consists of a large firm with many plants independently producing and marketing a wide variety of products. The plants are linked only through the coordinating decision of the headquarters. Although Groves' definition of a conglomerate does not seem to address the internal transfer environment postulated in our model, we note that the derivation in the next section of this chapter considers the dynamic team problem (internal transfer) as a series of static team problems (conglomerates). Therefore, Groves' analysis is applicable to our model with the restrictions we impose in the derivation of the optimal decentralized control model. It should be noted that our model excludes incentive schemes that are based on accounting measures which are affected by the choice of the transfer price. We do not address the issue of conflicting objectives which is the central problem of transfer pricing in a nonteam environment. Thus our assumption that incentive arrangements can be made to induce team behavior appears to be reasonable in view of Groves' work. The control technique that we introduced earlier can be extended to decentralized decision making. For the special case of two decision makers, eq. (3-1) can be restated as

$$J = E(x(N+1)-\bar{x}(N+1))'Q(N+1)(x(N+1)-\bar{x}(N+1))$$

$$+ \sum_{t=1}^{N} ((x(t)-\bar{x}(t))'Q(t)(x(t)-\bar{x}(t))$$

$$+ (u_1(t)-\bar{u}(t))'R_1(t)(u_1(t)-\bar{u}_1(t))$$

$$+ (u_2(t)-\bar{u}_2(t))'R_2(t)(u_2(t)-\bar{u}_2(t))) \qquad (4\text{-}5)$$

where J is the expected cost to the organization resulting from deviations between actual performance and planned performance. As before, individual decision makers are punished/rewarded according to deviations from targets for which they are held accountable. It should be noted that the "punishment" concept can be, and in most cases, would be transformed to a reward system to allow for behavioral implications. Although the concept of reward versus punishment is a moot point from an analytic perspective, the ultimate success of the actual control technique could be affected in large measure by the manner in which it is presented to the team members.

Derivation of Optimal Team Decision Policy

The formulation presented here is based on a combination of the results for the linear quadratic Gaussian problem assuming a one-step, delayed, information sharing pattern and Radner's static team problem with quadratic cost criterion. The theoretical approach closely follows

that taken by Speyer and Krainak (1979). A dynamic programming algorithm is applied starting at the terminal stage and proceeding backwards in time. At each stage the cost-to-go is determined, conditioned on the past centralized information assumed available to all decision makers. Minimizing this cost to go is essentially a static team problem where each decision maker is to make a decision based upon his present information (which only he has) and the past information shared by all the team members. The success of this procedure relies heavily on the ability to reproduce the quadratic cost functional form for the cost-to-go at each stage.

In dynamic programming the procedure is to start at the terminal stage and develop a recursion relationship operating backwards in time. The cost can be written as a sequence of nested conditional expectations as

$$\begin{aligned} J = \; & E\{E\{(x_1-\bar{x}_1)^T Q_1 (x_1-\bar{x}_1) + (u_1-\bar{u}_1)^T R_1 (u_1-\bar{u}_1) \\ & +\{E \ldots + \ldots E\{(x_{N+1}-\bar{x}_{N+1})^T Q_{N+1} \\ & (x_{N+1}-\bar{x}_{N+1})/I_{N+1}\}/\ldots\}/I_1\}\} \end{aligned} \tag{4-6}$$

The notation $E\{(\cdot)/I_i\}$ denotes the expectation operation conditioned on I_i. The nesting of expectation is done with respect to the shared information pattern and does not include the decentralized portion. In this

derivation we denote the time period as a subscript for clarity of presentation.

Define the cost-to-go function as the cost to go to the final stage from stage i given I_i denoted as

$$J(I_i) = E\{E\{(x_i-\bar{x}_i)^T Q_i (x_i-\bar{x}_i) + (u_i-\bar{u}_i)^T R_i (u_i \bar{u}_i)$$
$$+ E\{\ldots+\ldots E\{(x_{N+1} - \bar{x}_{N+1})^T Q_{N+1}$$
$$(x_{N+1}-\bar{x}_{N+1})/I_{N+1}\}/\ldots\}/I_i\}\} \tag{4-7}$$

where it is assumed that an admissible control policy sequence has occurred up to stage i-1. A recursion formula for $J(I_i)$ for all $i\varepsilon[1,N]$ is obtained directly from (4-6) and (4-7) as

$$J(I_i) = E\{(x_i-\bar{x}_i)^T Q_i (x_i-\bar{x}_i) + (u_i-\bar{u}_i)^T R_i$$
$$(u_i-\bar{u}_i) + J(I_{i+1})/I_i\} \tag{4-8}$$

This recursion formula plays a central role in the development of the one-step delayed information sharing pattern.

Determination of Cost-to-Go at Final Stage

From (4-7) the cost-to-go at state N+1 is

$$J(I_{N+1}) = E\{(x_{N+1}-\bar{x}_{N+1})^T Q_{N+1} (x_{N+1}-\bar{x}_{N+1})/I_{N+1}\}. \tag{4-9}$$

The expectation can be explicitly determined because the conditional probability density function of x_{N+1} given I_{N+1} is normal with mean $\hat{x}_{N+1/N}$ and covariance $P_{N+1/N}$; i.e.,

$$p(x_{N+1}/I_{N+1}) \sim N(\hat{x}_{N+1/N}, P_{N+1/N}) \tag{4-10}$$

where $\hat{x}_{i+1/i}$ is the conditional mean of the state at stage i+1 given the measurement history and $p_{i+1/i}$ is the error covariance in estimating the state at i+1 based on the measurement history up to i. From Kalman filtering theory, the conditional mean $\hat{x}_{i+1/i}$ is propagated sequentially by the update formula

$$\hat{x}_{i+1/i} = A_i\hat{x}_{i/i} + B_iu_i \tag{4-11}$$

where

$$\hat{x}_{i/i} = \hat{x}_{i/i-1} + k_i\nu_i$$

and the zero mean white noise process ν_i is called the innovations process and is

$$\nu_i = y_i - \hat{y}_i \tag{4-12}$$

where

$$\hat{y}_i = H_i\hat{x}_{i/i-1}$$

is the estimate of the measurement y_i. The variance of ν_i is

$$E\{\nu_i \nu_i^T\} = H_i P_{i/i-1} H_i^T + V_i = \Lambda_i$$

The Kalman gain k_i is defined as

$$k_i = P_{i/i-1} H_i^T \Lambda_i^{-1}.$$

The error variance is defined as

$$\begin{aligned} P_{i/i-1} &= E\{(x_i - \hat{x}_{i(i-1)})(x_i - \hat{x}_{i/i-1})^T\} \\ &= A_{i-1} P_{i-1/i-1} A_{i-1}^T + W_{i-1}. \end{aligned}$$

for convenience, define

$$e_i = x_i - \hat{x}_{i/i-1} \;;\; E\{e_i\} = 0,\; E\{e_i e_i^T\} = P_{i/i-1}$$

$$c_i = \hat{x}_{i/i-1} - \bar{x}_i.$$

Thus (4-9) becomes

$$\begin{aligned} J(I_{N+1}) &= E\{(e_{N+1} + c_{N+1})^T Q_{N+1} (e_{N+1} + c_{N+1})/I_{N+1}\} \\ &= E\{e_{N+1}^T Q_{N+1} e_{N+1} + c_{N+1}^T Q_{N+1} c_{N+1} \\ &\quad + 2e_{N+1}^T Q_{N+1} c_{N+1}/I_{N+1}\} \\ &= \operatorname{tr} Q_{N+1} P_{N+1/N} + (\hat{x}_{N+1/N} - \bar{x}_{N+1})^T Q_{N+1} \\ &\qquad (\hat{x}_{N+1/N} - \bar{x}_{N+1}) \end{aligned}$$

$$= \operatorname{tr} Q_{N+1}P_{N+1/N} + \hat{x}_{N+1/N}{}^T Q_{N+1}\hat{x}_{N+1/N}$$

$$- 2\bar{x}_{N+1}{}^T Q_{N+1}\hat{x}_{N+1/N} + \bar{x}_{N+1}{}^T Q_{N+1}\bar{x}_{N+1}$$

$$J(I_{N+1}) = K_{N+1} + \hat{x}_{N+1/N}{}^T Q_{N+1}\hat{x}_{N+1/N} - 2Z_{N+1}\hat{x}_{N+1/N} \qquad (4\text{-}13)$$

where:

$$K_{N+1} = \operatorname{tr} Q_{N+1}P_{N+1/N} + \bar{x}_{N+1}{}^T Q_{N+1}\bar{x}_{N+1}$$

$$Z_{N+1} = \bar{x}_{N+1}{}^T Q_{N+1}$$

Determination of Cost-to-Go From Stage N to Stage N+1

By using (4-8), the problem of finding the minimum value of J, J*, is obtained recursively by using the fundamental theorem in Meier, et al. (1971) which states that

$$J^* = \underset{u_i \forall i \in [1,N]}{\text{minimum}} E\{J(I_N)\} = \underset{u_i \forall i \in [1,N-1]}{\text{minimum}} E\{\min_{u_N} J(I_N)\} \qquad (4\text{-}14)$$

Note that $J(I_N)$ has been used for convenience to denote $J(I_N,u_N^j(I_N))$ $j=1,\ldots,k)$ where $u_N^j(I_N^j)$ for $j=1,\ldots,k$ is now to be determined by the minimization in (4-14) as

$$J^*(I_N) = \min_{u_N(I_N)} E\{(x_N-\bar{x}_N)^T Q_N(x_N-\bar{x}_N) + (u_N-\bar{u}_N)^T R_N(u_N-\bar{u}_N)$$

$$+ J(I_{N+1})/I_N\} \qquad (4\text{-}15)$$

$$J^*(I_N) = \min_{u_N(I_N)} E\{(e_N+c_N)^T Q_N (e_N+c_N) + (u_n-u_N)^T R_N (u_N-u_N)$$

$$+ K_{N+1} + \hat{x}_{N+1}^T Q_{N+1} \hat{x}_{N+1/N}$$

$$- 2Z_{N+1} \hat{x}_{N+1/N} / I_N\}$$

$$J^*(I_N) = \min_{u_N(I_N)} E\{\mathrm{tr}\, Q_N P_{N-1} + \hat{x}_{N/N-1}^T Q_N \hat{x}_{N/N-1}$$

$$- 2\bar{x}_x^T Q_N \hat{x}_{N/N-1} + \bar{x}_N^T Q_N \bar{x}_N + u_N^T R_N u_N$$

$$+ \bar{u}_N^T R_N \bar{u}_N - 2u_N^T R_N \bar{u}_N + K_{N+1}$$

$$+ (A_N \hat{x}_{N/N-1} + B_N u_N + A_k k_N \nu_N)^T Q_{N+1}$$

$$\cdot (A_N \hat{x}_{N/N-1} + B_N u_N + A_N k_N \nu_N)$$

$$- 2Z_{N+1}(A_N \hat{x}_{N/N+1} + B_N u_N + A_N k_N \nu_N)/I_N\}$$

$$J^*(I_N) = \min_{u_N(I_N)} E\{K_N + K_{N+1} + \bar{u}_N^T R_N \bar{u}_N + \hat{x}_{N/N-1}^T (Q_N + A_N^T Q_{N+1} A_N)$$

$$\hat{x}_{N/N-1} + u_N^T (B_N^T Q_{N+1} B_N + R_N) u_N$$

$$- 2(Z_N + Z_{N+1} A_N) \hat{x}_{N/N-1}$$

$$- 2u_N^T (R_N \bar{u}_N + B_N^T Z_{N+1}^T)$$

$$+ \nu_N^T k_N^T A_N^T Q_{N+1} A_N k_N \nu_N + 2u_N^T B_N^T Q_{N+1} A_N \hat{x}_{N/N-1}$$

$$+ 2\, u_N^T B_N^T Q_{N+1} A_N k_N \nu_N + 2\nu_N^T k_N^T A_N^T Q_{N+1} A_N \hat{x}_{N/N-1}$$

$$- 2\, \nu_N^T k_N^T A_N^T Z_{N+1}^T / I_N\} \tag{4-16}$$

define

$$\underline{K}_N = K_N + K_{N+1} + \bar{u}_N{}^T R_N \bar{u}_N$$

$$\underline{Q}_N = Q_N + A_N{}^T Q_{N+1} A_N$$

$$\underline{R}_N = B_N{}^T Q_{N+1} B_N + R_N$$

$$\underline{Z}_N = -[Z_N + Z_{N+1} A_N]$$

$$\underline{d}_N = -[R_N \bar{u}_N + B_N{}^T Z_{N+1}{}^T]$$

$$\underline{N}_N = k_N{}^T A_N{}^T Q_{N+1} A_N k_N$$

$$\underline{S}_N = B_N{}^T Q_{N+1} A_N$$

$$\underline{M}_N = B_N{}^T Q_{N+1} A_N k_N$$

$$\underline{Y}_N = k_N{}^T A_N{}^T Q_{N+1} A_N$$

$$\underline{a}_N = -[k_N{}^T A_N{}^T Z_{N+1}{}^T]$$

then (4-16) can be written as

$$\begin{aligned} J^*(I_N) = \underset{u_N(I_N)}{\text{Minimum}}\ E\{ & \underline{K}_N + \hat{x}_{N/N-1}{}^T \underline{Q}_N \hat{x}_{N/N-1} \\ & + u_N{}^T \underline{R}_N u_N + 2\underline{Z}_N \hat{x}_{N/N-1} \\ & + 2u_N{}^T \underline{d}_N + \nu_N{}^T \underline{N}_N \nu_N + 2u_N{}^T \underline{S}_N \hat{x}_{N/N-1} \\ & + 2u_N{}^T \underline{M}_N \nu_N + 2\nu_N{}^T \underline{Y}_N \hat{x}_{N/N+1} + 2\nu_N{}^T \underline{a}_N / I_N\}. \end{aligned} \tag{4-17}$$

Since $\hat{x}_{N/N-1}$ is measureable with respect to the σ-algebra generated by I_N, the random vector over which the expectation conditioned on I_N is taken, is ν_N. Note that since $u_N{}^i$ depends on $I_N{}^i$, $u_N{}^i$ depends explicitly on the local information $\nu_N{}^i$. The determination of $u_N{}^j(I_N{}^j)$ is a static decentralized team problem. Before proceeding to the static team problem, we define ϕ_N as

$$\phi_N = u_N{}^T \underline{R}_N u_N + 2u_N{}^T \underline{d}_N + 2u_N{}^T \underline{S}_N \hat{x}_{N/N-1}$$

$$+ 2u_N{}^T \underline{M}_N \nu_N + \nu_N{}^T \underline{M}_N \nu_N + 2\nu_N{}^T \underline{Y}_N \hat{x}_{N/N-1}$$

$$+ 2\nu_N{}^T \underline{a}_N. \tag{4-18}$$

Using (4-18), we rewrite (4-17) as

$$J^*(I_N) = \underline{K}_N + \hat{x}_{N/N-1}{}^T \underline{Q}_N \hat{x}_{N/N-1} + 2\underline{Z}_N \hat{x}_{N/N+1}$$

$$+ \min_{u_N} E\{\phi_N/I_N\}. \tag{4-19}$$

The static team problem is to minimize $J(I_N)$ with respect to the control function $u_N{}^j(I_N{}^j)$ where $I_N{}^j$ is defined as the union of I_N and $\nu_N{}^j$. The decision function can be written as

$$u_N{}^j(I_N{}^j) = u_N{}^j(\nu_N{}^j, \hat{x}_{N/N-1}). \tag{4-20}$$

Radner showed that for a quadratic cost criterion, this functional form is linear and can be expressed as

$$u_N^j(I_N^j) = D_N^j \nu_N^j + C_N^j(\hat{x}_{N/N+1}). \tag{4-21}$$

which satisfies a coupled set of stationary conditions.

To develop the stationary conditions, suppose that the decision functions of all but one of the team members are fixed. Then, a one-person minimization is performed by assuming that the fixed decision functions of the other team members are at their one-person minimum denoted as $\hat{u}_N^j(I_N^j)$. The one-person cost criterion is

$$E\{\phi_N^j(u_N^j)/I_N^j\} = E\{\phi_N(\hat{u}_N^1(I_N^1),\ldots,\hat{u}_N^{j-1}(I_N^{j-1}),$$
$$u_N^j, \hat{u}_N^{j+1}(I_N^{i+1}),\ldots,\hat{u}_N^k(I_N^k))\} \tag{4-22}$$

under proper conditions (Radner:1962), the operations of expectation and differentiation can be exchanged to give

$$\frac{\partial}{\partial u_N^j} E\{\phi_N^j(u_N^i)/I_N^j\} = E\left\{\frac{\partial(\phi_N^j/I_N^j)}{\partial u_N^j}\right\} = 0;$$
$$j=1,\ldots,k \tag{4-23}$$

Equation (4-23) results in the following K stationary conditions

$$E\{\frac{\partial}{\partial u_N^j}[u_N^T \underline{R}_N u_N + 2u_N^T \underline{d}_N + 2u_N^T \underline{S}_N \hat{x}_{N/N-1}$$
$$+ 2u_N^T \underline{M}_N \nu_N/I_N^j]\} = 0;\ j=1,\ldots,K \tag{4-24}$$

Thus, the first-order conditions become

$$E\{\sum_{l=1}^{K}(\underline{R}_N^{jl}u_N^{l}) + \underline{S}_N^{j}\hat{x}_{N/N-1} + \underline{d}_N^{j}$$

$$+ \underline{M}_N^{j}\nu_N/I_N^{j}\} = 0; \quad j=1,\ldots,K \qquad (4\text{-}25)$$

where

$(\_)_N^{j}$ denotes the j^{th} row of $(\_)_N$

note that

$$\underline{M}_N^{j}\nu_N = \sum_{l=1}^{K}\underline{M}_N^{jl}\nu_N^{l}. \qquad (4\text{-}26)$$

Rewriting (4-25) and (4-26) and considering (4-21) results in

$$E\{\sum_{l=1}^{K}(\underline{R}_N^{jl}D_N^{l} + \underline{M}_N^{jl})\nu_N^{l}/I_N^{j}\} + \sum_{l=1}^{K}\underline{R}_N^{jl}C_N^{l}$$

$$+ \underline{S}_N^{j}\hat{x}_{N/N-1} + \underline{d}_N^{j} = 0; \quad j=1,\ldots,K \qquad (4\text{-}27)$$

to evaluate expectations, recall

$$p(\nu_i^{j}) \sim N(0,\Lambda_i) \sim N(0,H_I^{j}P_{i/i-1}H_i^{jT} + V_i^{j})$$

note that

$$E\{\nu_i^{\ i}\nu_i^{\ jT}\} = E\{\nu^i\nu^{jT}\} = E\{(y_i^{\ i}-H_i^{\ i}\hat{x}_{i/i-1})$$

$$(y_i^{\ j}-H_i^{\ j}\hat{x}_{i/i-1})^T\}$$

$$E\{\nu_i^{\ i}\nu_i^{\ jT}\} = E\{[H_i^{\ i}(x_i-\hat{x}_{i/i-1})$$

$$(x_i-\hat{x}_{i/i-1})^T H_i^{\ jT} + H_i^{\ i}(x_i-\hat{x}_{i-1})\nu_i^{\ jT}$$

$$+ \nu_i^{\ i}(x_i-\hat{x}_{i-1})H_i^{\ JT} + \nu_i^{\ i}\nu_i^{\ jT}\}$$

$$E\{\nu_i^{\ i}\nu_i^{\ jT}\} = H_i^{\ i}P_{i/i-1}\ H_i^{\ jT} \tag{4-28}$$

therefore

$$p(\nu^i\nu^j) \sim N(0,L) \tag{4-29}$$

where

$$L = \left[\begin{array}{c|c} L^{ii} & L^{ij} \\ \hline L^{ji} & L^{jj} \end{array}\right] = \left[\begin{array}{c|c} H^iPH^{iT}+V^i & H^iPH^{jT} \\ \hline H^jPH^{iT} & H^jPH^{jT}+V^j \end{array}\right]$$

and the conditional density is (Jaswinski, p.45:1970);

$$p(\nu^i/\nu^j) \sim N(L^{ij}(L^{ij})^{-1}\nu^j,\ L^{ii}-L^{ij}(L^{jj})^{-1}L^{ji}) \tag{4-30}$$

Using (4-30) we define the following conditional means

$$E\{\nu_N^{\ j}/\nu_N^{\ j}\} = \nu_N^{\ j} \tag{4-31}$$

$$E\{\nu_N^{l}/\nu_N^{j} = L_N^{lj}(L^{jj})^{-1}\ \nu_N^{j}$$

$$= H_N^{l}P_{N/N-1}H_N^{jT}(H_N^{j}P_{N/N-1}H_N^{jT} + V_N^{j})^{-1}\nu_N^{j}$$

(4-32)

Combining (4-27), (4-31), and (4-32) and taking the expectation explicitly yields

$$(\underline{R}_N^{jj}D_N^{j} + \underline{M}_N^{jj})\nu_N^{j} + \sum_{\substack{l=1\\ l\neq j}}^{K} (\underline{R}_N^{jl}D_N^{l} + \underline{M}_N^{jl})$$

$$H_N^{l}P_{N/N-1}H_N^{jT}\Lambda_N^{j-1}\nu_N^{j}$$

$$+ \sum_{l=1}^{K} R_N^{jl}C_N^{l} + \underline{S}_N^{j}\hat{x}_{N/N-1} + \underline{d}_N^{j} = 0;\ j=1,\ldots,K$$

(4-33)

since ν_N^{j} is arbitrary

$$\underline{R}_N^{jj}D_N^{j} + \underline{M}_N^{jj} + \sum_{\substack{l=1\\ l\neq j}}^{K} (\underline{R}_N^{jl}D_N^{l} + \underline{M}_N^{jl})$$

$$H_N^{l}P_{N/N-1}H_N^{jT}\Lambda_N^{j-1} = 0;\ j=1,\ldots,K \quad (4\text{-}34)$$

and

$$\sum_{l=1}^{K} \underline{R}_N^{jl}C_N^{l} + \underline{S}_N^{j}\hat{x}_{N/N-1} + \underline{d}_N^{j} = 0;\ j=1,\ldots,K \quad (4\text{-}35)$$

rewriting (4-35), since $\underline{R}_N > 0$

$$C_N = -\underline{R}_N^{-1}[\underline{S}_N\hat{x}_{N/N-1} + \underline{d}_N] \tag{4-36}$$

thus,

$$u_N = D_N\nu_N + C_N \tag{4-37}$$

where u_N is the decision vector for the team as a whole at time N and D_N is the block diagonal matrix formed from the individual decision maker's innovations process gains where D_N is determined as the unique solution to (4-34) and C_N is given by (4-36). Inserting (4-37) into (4-19) yields

$$\begin{aligned}
J^*(I_N) = {} & \underline{K}_N + \hat{x}_{N/N-1}{}^T\underline{Q}_N\hat{x}_{N/N-1} + 2\underline{Z}_N\hat{x}_{N/N-1} \\
& + E\{(D_N\nu_N - \underline{R}_N^{-1}\underline{S}_N\hat{x}_{N/N-1} - \underline{R}_N^{-1}\underline{d}_N)^T \\
& \quad \underline{R}_N(D_N\nu_N - \underline{R}_N^{-1}\underline{S}_N\hat{x}_{N/N-1} - \underline{R}_N^{-1}\underline{d}_N) \\
& + 2(D_N\nu_N - \underline{R}_N^{-1}\underline{S}_N\hat{x}_{N/N-1} - \underline{R}_N^{-1}\underline{d}_N)^T\underline{d}_N \\
& + 2(D_N\nu_N - \underline{R}_N^{-1}\underline{S}_N\hat{x}_{N/N-1} - \underline{R}_N^{-1}\underline{d}_N)^T\underline{S}_N\hat{x}_{N/N-1} \\
& + 2(D_N\nu_N - \underline{R}_N^{-1}\underline{S}_N\hat{x}_{N/N-1} - \underline{R}_N^{-1}\underline{d}_N)^T\underline{M}_N\nu_N \\
& + \nu_N{}^T\underline{N}_N\nu_N + 2\nu_N{}^T\underline{Y}_N\hat{x}_{N/N-1} \\
& + 2\nu_N{}^T\,\underline{a}_N/I_N\}
\end{aligned} \tag{4-38}$$

$$J^*(I_N) = \underline{K}_N + \hat{x}_{N/N-1}{}^T \underline{Q}_N \hat{x}_{N/N-1} + 2\underline{Z}_N \hat{x}_{N/N-1}$$

$$+ E\{\nu_N{}^T D_N{}^T \underline{R}_N{}^{-1} D_N \nu_N + \hat{x}_{N/N-1}{}^T \underline{S}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N \hat{x}_{N/N-1}$$

$$+ \underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{d}_N + 2\underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N \hat{x}_{N/N-1}$$

$$-2\underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N \hat{x}_{N/N-1} - 2\underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{d}_N$$

$$- 2\hat{x}_{N/N-1}{}^T \underline{S}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N \hat{x}_{N/N-1} - 2\underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N \hat{x}_{N/N-1}$$

$$+ 2\nu_N{}^T D_N{}^T \underline{M}_N \nu_N + \nu_N{}^T \underline{N}_N \nu_N / I_N\} \qquad (4\text{-}39)$$

$$J^*(I_N) = \underline{K}_N + \hat{x}_{N/N-1}{}^T \underline{Q}_N \hat{x}_{N/N-1} + 2\underline{Z}_N \hat{x}_{N/N-1}$$

$$+ \operatorname{tr}\ (D_N{}^T \underline{R}_N{}^{-1} D_N + 2\underline{M}_N{}^T D_N + \underline{N}_N) \Lambda_N$$

$$- \hat{x}_{N/N-1}{}^T \underline{S}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N \hat{x}_{N/N-1} - \underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{d}_N$$

$$- 2\underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N \hat{x}_{N/N-1} \qquad (4\text{-}40)$$

define:

$$\bar{K}_N = \underline{K}_N + \operatorname{tr}\ (D_N{}^T \underline{R}_N{}^{-1} D_N + 2\underline{M}_N{}^T D_N + \underline{N}_N)$$

$$\Lambda_N - \underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{d}_N$$

$$\bar{Q}_N = \underline{Q}_N - \underline{S}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N$$

$$\bar{Z}_N = \underline{d}_N{}^T \underline{R}_N{}^{-1} \underline{S}_N - \underline{Z}_N$$

thus we rewrite (4-40) as

$$J^*(I_N) = \bar{K}_N + \hat{x}_{N/N-1}{}^T\bar{Q}_N\hat{x}_{N/N-1} - 2\underline{\bar{Z}}_N\hat{x}_{N/N-1} \qquad (4\text{-}41)$$

Note that $J^*(I_N)$ given by (4-41) is functionally similar to $J(I_{N+1})$ given by (4-13).

Determination of Cost-to-Go From Stage N-1 to Final Stage

$$J^*(I_{N-1}) = \min_{u_{N-1}(I_{N-1})} E\{(x_{N-1}-\bar{x}_{N-1})^T Q_{N-1}(x_{N-1}-\bar{x}_{N-1})$$

$$+ (u_{N-1}-\bar{u}_{N-1})^T R_{N-1}(u_{N-1}-\bar{u}_{N-1}) + J(I_N)/I_{N-1}\} \qquad (4\text{-}42)$$

$$J^*(I_{N-1}) = \bar{K}_{N-1} + \hat{x}_{N-1/N-2}{}^T\bar{Q}_{N-1}\hat{x}_{N-1/N-2} + 2\bar{Z}_{N-1}\hat{x}_{N-1/N-2}$$

$$+ \min_{u_{N-1}} E\{u_{N-1}{}^T\bar{R}_{N-1}u_{N-1} + 2u_{N-1}{}^T\bar{d}_{N-1}$$

$$+ \nu_{N-1}{}^T\bar{N}_{N-1}\nu_{N-1} + 2u_{N-1}{}^T\bar{S}_{N-1}\hat{x}_{N-1/N-2}$$

$$+ 2u_{N-1}{}^T\bar{M}_{N-1}\nu_{N-1} + \nu_{N-1}{}^T\underline{Y}_{N-1}\hat{x}_{N-1/N-2}$$

$$+ 2\nu_{N-1}{}^T\bar{a}_{N-1}/I_{N-1}\} \qquad (4\text{-}43)$$

where

$$\bar{K}_{N-1} = K_{N-1} + \bar{K}_N + \bar{u}_{N-1}{}^T R_{N-1}\bar{u}_{N-1}$$

$$\bar{Q}_{N-1} = Q_{N-1} + A_{N-1}{}^T\bar{Q}_N A_{N-1}$$

$$\bar{R}_{N-1} = R_{N-1} + B_{N-1}{}^T\bar{Q}_N B_{N-1}$$

$$\bar{Z}_{N-1} = -[Z_{N-1} + \bar{Z}_N A_{N-1}]$$

$$\bar{d}_{N-1} = -[R_{N-1}\bar{u}_{N-1} + B_{N-1}{}^{T}\bar{Z}_{N}{}^{T}]$$

$$\bar{N}_{N-1} = k_{N-1}{}^{T}A_{N-1}{}^{T}\bar{Q}_{N}A_{N-1}k_{N-1}$$

$$\bar{S}_{N-1} = B_{N-1}{}^{T}\bar{Q}_{N}A_{N-1}$$

$$\bar{M}_{N-1} = B_{N-1}{}^{T}\bar{Q}_{N}A_{N-1}k_{N-1}$$

$$\bar{Y}_{N-1} = k_{N-1}A_{N-1}{}^{T}\bar{Q}_{N}A_{N-1}$$

$$\bar{a}_{N-1} = -[k_{N-1}{}^{T}A_{N-1}{}^{T}\bar{Z}_{N}{}^{T}]$$

Note that (4-43) is of the same functional form as (4-17) which indicates that Radner's static team problem must be solved again for $J^*(I_{N-1})$ which results in

$$u_{N-1} = D_{N-1}\nu_{N-1} + C_{N-1} \tag{4-44}$$

where D_{N-1} is determined by

$$\bar{R}_{N-1}{}^{jj}D_{N-1}{}^{j} + \bar{M}_{N-1}{}^{jj} + \sum_{\substack{l=1 \\ l\neq j}}^{K} (\bar{R}_{N-1}{}^{jl}D_{N-1}{}^{l} + \bar{M}_{N-1}{}^{jl})$$
$$H_{N-1}{}^{l}P_{N-1/N-2}H_{N-1}{}^{jT}\Lambda_{N-1}{}^{j-1} = 0; \; j=1,\ldots,K \tag{4-45}$$

and

$$C_{N-1} = -\bar{R}_{N-1}{}^{-1}[\bar{S}_{N-1}\hat{x}_{N-1/N-2} + \bar{d}_{N-1}] \tag{4-46}$$

thus $J^*(I_{N-1})$ becomes

$$J^*(I_{N-1}) = \bar{\bar{K}}_{N-1} + \hat{x}_{N-1/N-2}{}^T\bar{\bar{Q}}_{N-1}\hat{x}_{N-1/N-2}$$

$$-2\bar{\bar{Z}}_{N-1}\hat{x}_{N-1/N-2} \qquad (4\text{-}47)$$

where

$$\bar{\bar{K}}_{N-1} = \bar{K}_{N-1} + tr\ (D_{N-1}{}^T\bar{R}_{N-1}{}^{-1}D_{N-1}$$

$$+ 2\bar{M}_{N-1}^T D_{N-1} + \bar{N}_{N-1})\Lambda_{N-1}$$

$$- \bar{d}_{N-1}{}^T\bar{R}_{N-1}{}^{-1}\bar{d}_{N-1}$$

$$\bar{\bar{Q}}_{N-1} = \bar{Q}_{N-1} - \bar{S}_{N-1}\bar{R}_{N-1}{}^{-1}\bar{S}_{N-1}$$

$$\bar{\bar{Z}}_{N-1} = \bar{d}_{N-1}{}^T\bar{R}_{N-1}{}^{-1}\bar{S}_{N-1} - \bar{Z}_{N-1}.$$

Since $J^*(I_{N-1})$ given by (4-47) is functionally similar to $J^*(I_N)$ and $J(I_{N+1})$, the general recursion relationships in going from stage i+1 to stage i can be stated by appealing to an induction argument. That is, the results in going from N+1 to N hold in going from state i+1 to i if i replaces N. By induction, the optimal decentralized control policy using the one-step delayed information sharing pattern at stage i is

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$$u_K(\nu_K, \hat{x}_{K(K-1)}) = D_K \nu_K - [R_K + B_K{}^T \tilde{Q}_{K+1} B_K]^{-1}$$

$$[B_K{}^T \tilde{Q}_{K+1} A_K \hat{x}_{K/K-1} - R_K \bar{u}_K - B_K{}^T \tilde{Z}_{K+1}{}^T] \quad (4\text{-}48)$$

where D_K is determined by:

$$(R_K + B_K{}^T \tilde{Q}_{K+1} B_K)^{jj} D_K{}^j + (B_K{}^T \tilde{Q}_{K+1} A_K k_K)^{jj}$$

$$+ \sum_{\substack{i=1 \\ l \neq j}}^{K} \{ [(R_K + B_K{}^T \tilde{Q}_{K+1} B_K)^{jl} D_K{}^l + (B_K{}^T \tilde{Q}_{K+1} A_K k_K)^{jl}]$$

$$H_K{}^l P_{K/K-1} H_K{}^{j^T} \Lambda_K{}^{j-1} \} = 0, j=1,\ldots,K \quad (4\text{-}49)$$

and

$$\tilde{Q}_K = Q_K + A_K{}^T \tilde{Q}_{K+1} A_K - (B_K{}^T \tilde{Q}_{K+1} A_K)^T$$

$$(R_K + B_K{}^T \tilde{Q}_{K+1} B_K)^{-1} (B_K{}^T \tilde{Q}_{K+1} A_K) \quad (4\text{-}50)$$

$$\tilde{Q}_{N+1} = Q_{N+1} \quad (4\text{-}51)$$

$$\tilde{Z}_K = Z_K + \tilde{Z}_{K+1} A_K - (R_K \bar{u}_K + B_K \tilde{Z}_{K+1}{}^T)^T$$

$$(R_K + B_K{}^T \tilde{Q}_{K+1} B_K)^{-1} (B_K{}^T \tilde{Q}_{K+1} A_K) \quad (4\text{-}52)$$

$$\tilde{Z}_{N+1} = Z_{N+1} = \bar{x}_{N+1}{}^T Q_{N+1} \quad (4\text{-}53)$$

Equations (4-48) - (4-53) represent the generalized optimal control policy for a mean deviation quadratic penalty function. These control theory results extend prior

results in the literature for a two-person team to encompass a general n-person team setting. In addition, the performance measure allows the optimal control law to be a function of predetermined state and control objectives, or targets. It should be noted that the unpublished paper by Speyer and Krainak (1979) has solved a K-person linear exponential-Gaussian control problem where the controls are penalized over time and the final state is penalized.

Our general results can now be applied to the two-member team model of this dissertation. However, we note that the control theory results represented by eqns. (4-43) - (4-48) can be applied to problems involving both time-varying parameters in the system and observation model in addition to considering time varrying error variance in both the state and the observations.

Application of General Results to Two-Person Team

For a two-person team equations (4-48) - (4-53) can be expressed as

$$u_K = D_K \nu_K + E_K \hat{x}_{K/K-1} + F_K \tag{4-54}$$

where

$$D_K = [D^{1K} D^{2K}]^T$$

and can be determined as the solution to the following set of coupled equations:

$$
\begin{aligned}
&[B_K^{1T}\tilde{Q}_{K+1}B_K^{1} + R_K^{1}]\, D_K^{1}[H_K^{1}P_{K/K-1}H_K^{1T} + V_K^{1}] \\
&\quad + B^{1T}\tilde{Q}_{K+1}B_K^{2}D_K^{2}H_K^{2}P_{K/K-1}H_K^{1T} \\
&\quad = -B_K^{1T}\tilde{Q}_{K+1}A_KP_{K/K-1}H_K^{1T}
\end{aligned}
\tag{4-55}
$$

$$
\begin{aligned}
&B_K^{2T}\tilde{Q}_{K+1}B_K^{1}D_K^{1}H_K^{1}P_{K/K-1}H_K^{2T} + [B_K^{2T}\tilde{Q}_{K+1}B_K^{2}+R_K^{2}] \\
&\quad D_K^{2}[H_K^{2}P_{K/K-1}H_K^{2T}+V_K^{2}] = -B_K^{2T}\tilde{Q}_{K+1}A_KP_{K/K-1}H_K^{2T}
\end{aligned}
\tag{4-56}
$$

in addition

$$
E_K = \begin{bmatrix} E_K^{1} \\ E_K^{2} \end{bmatrix} = - \begin{bmatrix} R_K+B_K^{1T}\tilde{Q}_{K+1}B_K^{1} & B_K^{1T}\tilde{Q}_{K+1}B_K^{2} \\ B_K^{2T}\tilde{Q}_{K+1}B_K^{1} & R_K^{2}+B_K^{2T}\tilde{Q}_{K+1}B_K^{2} \end{bmatrix}^{-1}
\begin{bmatrix} B_K^{1T}\tilde{Q}_{K+1}A_K \\ B_K^{2T}\tilde{Q}_{K+1}A_K \end{bmatrix}
\tag{4-57}
$$

and

$$F_K = \begin{bmatrix} F_K^{\ 1} \\ F_K^{\ 2} \end{bmatrix} = \begin{bmatrix} R_K^{\ 1}+B_K^{\ 1T}\tilde{Q}_{K+1}B_K^{\ 1} & B_K^{\ 1T}\tilde{Q}_{K+1}B_K^{\ 2} \\ B_K^{\ 2T}\tilde{Q}_{K+1}B_K^{\ 1} & R_K^{\ 2}+B_K^{\ 2T}\tilde{Q}_{K+1}B_K^{\ 2} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} R_K^{\ 1}\bar{u}_K^{\ 1}+B_K^{\ 1T}\tilde{z}_{K+1}^{\ T} \\ R_K^{\ 2}\bar{u}_K^{\ 2}+B_K^{\ 2T}\tilde{z}_{K+1}^{\ T} \end{bmatrix} \qquad (4\text{-}58)$$

The application of equations (4-54) through (4-58) to the decentralized model of the firm developed in this dissertation results in further simplifications due to our assumption of time-invariant parameters and statistics.

Optimal Control Policy for Decentralized Model of the Firm

The optimal decision for the Production Division can be expressed as

$$u_1(t) = D_1(t)\nu(t) + E_1(t)\hat{x}(t/t-1) + F_1(t) \qquad (4\text{-}59)$$

$$u_1(t) = D_1(t)y_1(t) + [E_1(t) - D_1(t)H_1(t)]$$
$$\hat{x}(t/t-1) + F_1(t) \qquad (4\text{-}60)$$

Similarly, the optimal decision policy for the Marketing Division can be expressed as

$$u_2(t) = D_2(t)\nu(t) + E_2(t)\hat{x}(t/t-1) + F_1(t) \qquad (4\text{-}61)$$

$$u_2(t) = D_2(t)y_2(t) + [E_2(t) - D_2(t)H_2(t)]$$

$$\hat{x}(t/t-1) + F_2(t) \tag{4-62}$$

where $D_1(t)$ and $D_2(t)$ can be found as the unique solution to the following set of coupled equations:

$$[B_1{}^T\tilde{Q}(t+1)B_1 + R_1]D_1(t)[H_1P(t/t-1)H_1{}^T+V_1]$$

$$+ B_1{}^T\tilde{Q}(t+1)B_2D_2(t)H_2P(t/t-1)H_1{}^T$$

$$= -B_1{}^T\tilde{Q}(t+1)AP(t/t-1)H_1{}^T \tag{4-63}$$

$$[B_2{}^T\tilde{Q}(t+1)B_1D_1(t)H_1P(t/t-1)H_2{}^T]$$

$$+ [B_2{}^T\tilde{Q}(t+1)B_2+R_2]D_2(t)$$

$$[H_2P(t/t+1)H_2{}^T+V_2]$$

$$= -B_2{}^T\tilde{Q}(t+1)AP(t/t-1)H_2{}^T \tag{4-64}$$

further

$$\begin{bmatrix} E_1(t) \\ E_2(t) \end{bmatrix} = - \begin{bmatrix} R_1+B_1{}^T\tilde{Q}(t+1)B_1 & B_1{}^T\tilde{Q}(t+1)B_2 \\ B_2{}^T\tilde{Q}(t+1)B_1 & R_2+B_2{}^T\tilde{Q}(t+1)B_2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} B_1{}^T\tilde{Q}(t+1)A \\ B_2{}^T\tilde{Q}(t+1)A \end{bmatrix} \tag{4-65}$$

and

$$\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} = - \begin{bmatrix} R_1+B_1{}^T\tilde{Q}(t+1)B_1 & B_1{}^T\tilde{Q}(t+1)B_2 \\ B_2{}^T\tilde{Q}(t+1)B_1 & R_2+B_2{}^T\tilde{Q}(t+1)B_2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} R_1\bar{u}_1(t)+B_1{}^T\tilde{Z}(t+1)^T \\ R_2\bar{u}_2(t)+B_2{}^T\tilde{Z}(t+1)^T \end{bmatrix} \quad (4\text{-}66)$$

The theoretical results (eqns. (4-60) and (4-62)) provide us with an intuitively appealing decision process. We note that the optimal decisions are a weighted average of the current private information available to the decision maker plus information based on past actions and the current targets (or objectives) that have been established. Therefore, the decision maker considers all three of these sources to arrive at an optimal decision. The relative importance of private information versus historical information versus current objectives is determined by the system parameters, the observation parameters and the performance function weighting parameters.

Further analysis of the optimal decision policy characteristics is limited due to the theoretical nature of this dissertation. However, certain structural constraints currently imposed on the model do not appear to be atypical of a large number of actual decentralized

organizations. Additional restrictions are necessary to evaluate the effect of a change in transfer pricing policy. Thus, further conclusions reached in this dissertation must be limited to the current model as have developed and extensions to other models should be performed with caution.

Incentive Arrangement

Incentive arrangements are incorporated into our model through the performance measure function weighting matrices, Q, R_1 and R_2. The specific relationship is recalled as

$$\begin{aligned} J = E\ & (x(N+1)-\bar{x}(N+1))'Q(N+1)(x(N+1)-\bar{x}(N+1)) \\ & + \sum_{t=1}^{N} ((x(t)-\bar{x}(t))'Q(t)(x(t)-\bar{x}(t)) \\ & + (u_1(t)-\bar{u}_1(t))'R_1(u_1(t)-\bar{u}_1(t)) \\ & + (u_2(t)-\bar{u}_2(t))'R_2(u_2(t)-\bar{u}_2(t))) \qquad (4\text{-}67) \end{aligned}$$

where the objective of the team is to minimize the expected "cost" function, J. We observe that each variable can be isolated by appropriate definitions of the weighting matrices. For example, we observe that the weighting matrix Q can be used to affect the degree to which the decision makers attempt to achieve predetermined

values of the state variables where we defined the state as

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} p_r \\ p_T \\ b_2 \\ h_{TM} \\ h_{TP} \end{bmatrix} = \begin{bmatrix} \text{Raw material price} \\ \text{Transfer price} \\ \text{Demand parameter} \\ \text{Marketing inventory} \\ \text{Production inventory} \end{bmatrix} . \tag{4-68}$$

It was noted earlier that the decision makers have no direct control over the raw material price, the transfer price or the demand parameter and we would not expect to observe incentive arrangements regarding these variables. For the current model we would expect an incentive arrangement to result in a weighting matrix represented as

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_1 & 0 \\ 0 & 0 & 0 & 0 & Q_2 \end{bmatrix} \tag{4-69}$$

where Q_1 and Q_2 represent the incentives with regard to desired levels of inventory/backlog for the marketing and production divisions respectively. A similar

observation with respect to the control variables u_1 and u_2 which we defined as

$$u_1 = q_r = \text{production decision}$$

$$u_2 = \begin{bmatrix} q_M \\ p_f \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} \text{internal transfer decision} \\ \text{pricing decision} \end{bmatrix} \qquad (4\text{-}70)$$

leads us to anticipate the following structural forms:

$$R_1 = R_{11} \qquad (4\text{-}71)$$

$$R_2 = \begin{bmatrix} R_{21} & 0 \\ 0 & R_{22} \end{bmatrix} \qquad (4\text{-}72)$$

where Q_1, Q_2, R_{11}, and R_{22} represent the various incentive arrangements. It is reasonable to expect that the state and control variables may not be considered equally important to the firm as a whole. In addition, differences in unit measurements would indicate that a percentage deviation scheme might be more desirable than an absolute deviation philosophy. Unfortunately, the theoretical nature of this dissertation does not lend itself to detailed analysis with regard to alternative incentive arrangements although this particular aspect of the control model would consistute a significant portion of an

empirical application of the model. To facilitate further analysis in this study, we "weight" the target variables equally based on absolute deviation; that is, $Q_1 = Q_2 = R_{11} = R_{21} = R_{22} = 1$. We note that this assumption is not essential to the following analysis but is made merely as a mathematical convenience.

Information Structure

For illustrative purposes we assume the decentralized information structure is such that the production division receives current information concerning the raw material price, the transfer price and its own inventory/backlog status. Similarly, the marketing division receives current information concerning the transfer price and its own inventory/backlog status. This assumption posits a situation that would represent an expected lower bound on current information availability; i.e., we would expect the production decision maker to have access to current raw material prices, his current inventory position and the current transfer price. We would also envision the marketing decision maker to have access to his current inventory position and the current transfer price as a minimum. This information structure is captured in the model by defining the following information matrices:

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} . \tag{4-73}$$

We note that the production decision maker has current information concerning raw material prices and production inventory that is not available to the marketing division, while the marketing division has current information concerning their inventory position that is not available to the production decision maker. Further, both decision makers are aware of the current existing transfer price.

Transfer Pricing Policy Analysis

To facilitate analysis with respect to the impact of changes in the transfer pricing policy on our decentralized control model of the firm, we have posited an incentive arrangement and a specific information structure. We make one additional assumption with regard to the stochastic nature of the problem and assume that the stochastic parameters are zero-mean and unity variance random variables. This assumption does not affect the generality of the analysis and is made for mathematical

convenience. We restate eqns. (4-60) and (4-62) for reference as

$$u_1(t) = D_1(t)y_1(t) + (E_1(t)-D_1(t)H_1(t))\hat{x}(t/t-1) + F_1(t) \tag{4-74}$$

$$u_2(t) = D_2(t)y_2(t) + (E_2(t)-D_2(t)H_2(t))\hat{x}(t/t-1) + F_2(t) \tag{4-75}$$

and note that $F_1(t)$ and $F_2(t)$ are not dependent on the transfer price. We can determine the effect of transfer pricing changes on the optimal decentralized decisions by an analysis of

$$D_1(t)y_1(t) + (E_1(t) - D_1(t)H_1(t))\hat{x}(t/t-1) \tag{4-76}$$

to determine the impact of a change in the transfer price on the production decision. Similarly, we will be able to evaluate the impact of a transfer pricing change on the pricing decision and the internal exchange decision by examining

$$D_2(t)y_2(t) + (E_2(t)-D_2(t)H_2(t))\hat{x}(t/t-1). \tag{4-77}$$

We observe that the only time-varying matrix involved is $\tilde{Q}(t)$, whose solution is given by the discrete Riccati eqns. (4-50) and (4-51). To determine the impact of a change in transfer price on the optimal

decentralized decision policy we compute u(N) where $\tilde{Q}(N+1) = Q(N+1)$.

To determine $D_1(N)$ and $D_2(N)$ we compute the following matrices:

$$B_1'Q(N+1)B_1 + R_1 = 2$$

$$H_1P(N/N-1)H_1' + V_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B_1'Q(N+1)B_2 = [-1 \quad 0]$$

$$H_2P(N/N-1)H_1' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_1'Q(N+1)AP(N/N-1)H_1' = [0 \quad 0 \quad 1]$$

$$B_2'Q(N+1)B_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$H_1P(N/N-1)H_2' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_2'Q(N+1)B_2 + R_2 = \begin{bmatrix} 3 & b_1 \\ b_1 & (1+b_1^2) \end{bmatrix}$$

$$H_2P(N/N-1)H_2' + V_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$B_2'Q(N+1)AP(N/N-1)H_2' = \begin{bmatrix} 0 & 0 \\ 0 & b_1 \end{bmatrix} .$$

Substituting these values into eqs. (4-63) and (4-64) and solving the resulting set of coupled linear equations simultaneously results in the explicit solution of D(N) as follows:

$$D_1(N) = [0 \quad 0 \quad -1/4] \tag{4-78}$$

$$D_2(N) = \begin{bmatrix} 0 & 1/6 \\ 0 & -b_1/(2b_1^2 +3) \end{bmatrix}. \tag{4-79}$$

Further algebra and redefinition of constant terms results in the following:

$$D_1(N) = [0 \quad 0 \quad K_1] \tag{4-80}$$

$$D_2(N) = \begin{bmatrix} 0 & K_2 \\ 0 & K_3 \end{bmatrix} \tag{4-81}$$

$$E_1(N)-D_1(N)H_1 = [0 \quad 0 \quad K_4 \quad K_5 \quad K_6] \tag{4-82}$$

$$E_2(N)-D_2(N)H_2 = \begin{bmatrix} 0 & 0 & K_7 & K_8 & K_9 \\ 0 & 0 & K_{10} & K_{11} & K_{12} \end{bmatrix} \tag{4-83}$$

where K_i; $i=1,2,\ldots,12$ are known constants.

Combining eqns. (4-76) and (4-80) - (4-83) provides the information needed to explicitly determine the impact of the transfer price on the production decision as

$$u_1(N) \alpha [0 \;\; 0 \;\; K_1] \quad [p_r \;\; p_T \;\; h_{TP}]^T$$
$$+ \; [0 \;\; 0 \;\; K_4 \;\; K_5 \;\; K_6] \; [p_r \;\; p_T \;\; b_2 \;\; h_{TM} \;\; h_{TP}]^T. \quad (4\text{-}84)$$

We note that the optimal production decision is not dependent upon the actual transfer price in our model. Similarly, the marketing division's decision process is partially represented by rewriting eqn. (4-77) as

$$u_2(N) \alpha \begin{bmatrix} 0 & K_2 \\ 0 & K_3 \end{bmatrix} \begin{bmatrix} p_T \\ h_{TM} \end{bmatrix} + \begin{bmatrix} 0 & 0 & K_7 & K_8 & K_9 \\ 0 & 0 & K_{10} & K_{11} & K_{12} \end{bmatrix}$$
$$\cdot \; [p_r \;\; p_T \;\; b_2 \;\; h_{TM} \;\; h_{TP}]^T. \quad (4\text{-}85)$$

Again we observe that both the optimal pricing decision and the optimal internal product transfer decision are independent of the transfer price. Since the functional form of the optimal decision rule is repeated for all periods the analysis regarding the transfer pricing for period N can be extended for all decision periods. The implication of the above analysis is that, under the assumptions inherent in the research model,

the establishment of transfer pricing policy does not affect the optimal decentralized team decision making policy. The transfer pricing decision was removed from the control of the decentralized decision maker and the transfer price was assumed to be an exogenous variable which is generated by a first-order Markov process. Thus, under the restrictive conditions of the model (i.e., a team setting coupled with an exogenous transfer price), we observe that neither decision maker uses information concerning the transfer price and thus the determination of the transfer pricing policy is not important with respect to his optimal actions. Recall that the individual decision maker has knowledge of the impact of his decisions on the firm as a whole. In this setting, the decision maker would realize that the transfer price is an internal mechanism for the firm and as such it has no impact on the overall objectives of the firm in a team setting. This result holds for both one-period and multi-period analyses since the dynamic results we have generated can be easily reduced to a single-period analysis by discarding the time argument. This result can be anticipated by the realization that the decision makers are attempting to achieve corporate objectives which are derived from an overall perspective based on external market conditions. The internal transfer price

is merely a means for liquidity transference and should not affect the performance of the organization as a whole; that is, the optimal decentralized team decision making policy is independent of transfer pricing policy and the determination of an "optimal" transfer pricing policy should not be based on its effect on the optimal decentralized decision maker's actions.

CHAPTER V

SUMMARY AND DIRECTIONS FOR FUTURE RESEARCH

Summary

This dissertation has addressed the unique aspects involved in controlling a decentralized organization. We have integrated the concepts of modern control theory with the concepts of team theory to develop an optimal control policy. This policy was then applied to the conceptual framework developed for the analysis of decentralized decision making.

The main finding of this study is that the transfer price involved in the interdivisional exchange of goods or services does not affect the decentralized decision maker's actions. In a team setting we showed that the optimal decentralized decision-making policy is not dependent on the transfer price. If the operation of a transfer pricing system is costly to the organization as a whole, our research has shown that, for optimal decision making in a team setting, transfer pricing may be an ineffective decision-making tool. In this setting we would not expect to see a transfer pricing system used for decentralized decision making. If a team setting

exists in a decentralized organization, through some incentive system, the expenses involved with a transfer pricing system may be avoided and a resultant increase in efficiency could be realized. Although this result does not provide a procedure for determining the "optimal" transfer pricing policy, it does indicate that transfer pricing policy decisions should not be based on their impact on optimal decentralized decision maker's actions. However, this result was conditioned on two restrictive assumptions in the dissertation development. The first of these involved treating the organization as if it is a team. If this assumption is discarded and the transfer price becomes a mechanism to improve the position of one division at the expense of another, then the results of this dissertation would not be applicable.

The second assumption considered the transfer price itself as an exogenous variable generated as a first-order Markov process. The effect of removing the transfer pricing decision from the control of the decentralized producer of the transferred product was not investigated in this research study. It is not readily apparent what impact this assumption would make on the dissertation findings.

Directions for Future Research

The focus throughout this dissertation has been a theoretical one. However, the framework developed provides a means to evaluate the model empirically. Future research could branch in at least two complimentary directions. First, the framework itself can be refined by extending it to address parameter uncertainty. Second, empirical research could be performed in an econometric sense to determine how the model's decisions compare with actual decisions made by decentralized managers. Further analysis that considers the transfer price as a decision variable under the control of a decentralized decision maker needs to be performed. The results of this analysis would provide insight to the robustness of the findings of this dissertation.

An interesting empirical question lies unanswered concerning the main premise of this dissertation; i.e., an organiztion can be considered as a team which implies that manager/organizational goal conflicts are considered as higher order effects. In general, there appears to be a wide field open for research into the application of modern control theory to practical affairs in real organizations.

APPENDICES

APPENDIX A

STATE-SPACE REPRESENTATION OF A DYNAMIC SYSTEM

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STATE-SPACE REPRESENTATION OF A DYNAMIC SYSTEM[12]

The state-space description of a system has been used exclusively in modern control and systems theory. Its use derived from the motivation to represent any physical system by a number of first-order differential/ difference equations that relate an equal number of variables. If at any given time, the numerical values of these variables are known, the state of the system is completely specified, and if future inputs to the system are also known, the state of the system at any future time is also specified. Using the state-space approach an n order difference equation can be described as n first order difference equations which can be written compactly as

$$x(t+1) = f(x(t),u(t),t) \qquad \text{(A-1)}$$

where $x(t)$ is an n dimensional state vector, f is an n dimensional vector-valued function and $u(t)$ is an r dimensional control vector (or decision vector).

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[12]The material in this appendix is based on Pindyck (1973).

The advantage of describing a system by (A-1) is that it gives us a set of variables that completely determine the state and future behavior of the system. If at any time, $t=0$, the state $x(t=0)$ and all present and future values of the control $u(t)$ are known, then we can completely determine the state of the system $x(t)$ for any future time, t. The time path of the state vector is called the state trajectory. For a given system this will be determined by the initial state and the time path of the control vector (the control trajectory).

APPENDIX B

DERIVATION OF OPTIMAL CENTRALIZED PLANNING POLICY

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DERIVATION OF OPTIMAL CENTRAL PLANNING POLICY

<u>Problem</u>

<u>Minimize</u>:

$$J = (1/2) \sum_{t=0}^{N-1} [X^T(t)QX(t) + U^T(t)RU(t)$$

$$+ X^T(t)SU(T) + C'] + (1/2)X^T(N)QX(N) \qquad \text{(B-1)}$$

<u>Subject to</u>:

<u>Dynamics</u>

$$X(t+1) - X(t) = AX(t) + BU(t)$$

<u>Constraint</u>

$$CX(t) + DU(t) = 0$$

<u>Boundary Conditions</u>

$$X(t=0) = X(0)$$

$$t_f = N.$$

let:

$$L(x(t),u(t),t) = (1/2)X^T(t)QX(t) + (1/2)U^T(t)RU(t)$$

$$+ (1/2)X^T(t)SU(t) + (1/2)C$$

$$K(X(N)) = (1/2)X^T(N)QX(N)$$

APPENDIX B

DERIVATION OF OPTIMAL CENTRAL PLANNING POLICY

<u>Problem</u>

<u>Minimize</u>:

$$J = (1/2) \sum_{t=0}^{N-1} [X^T(t)QX(t) + U^T(t)RU(t)$$

$$+ X^T(t)SU(T) + C'] + (1/2)X^T(N)QX(N) \qquad \text{(B-1)}$$

<u>Subject to</u>:

<u>Dynamics</u>

$$X(t+1) - X(t) = AX(t) + BU(t)$$

<u>Constraint</u>

$$CX(t) + DU(t) = 0$$

<u>Boundary Conditions</u>

$$X(t=0) = X(0)$$

$$t_f = N.$$

let:

$$L(x(t),u(t),t) = (1/2)X^T(t)QX(t) + (1/2)U^T(t)RU(t)$$

$$+ (1/2)X^T(t)SU(t) + (1/2)C$$

$$K(X(N)) = (1/2)X^T(N)QX(N)$$

$$f(x(t),u(t),t) = AX + BU$$

$$\rho(x(t),u(t),t) = CX + DU$$

from eq. (2-8)

$$\partial H/\partial U|_{*} = 0 = RU^*(t) + (1/2)S^TX^*(t) + B^Tp^*(t+1)$$
$$- D^Tu^*(t+1)$$

thus

$$u^*(t) = -R^{-1}[(1/2)S^TX^*(t) + B^Tp^*(t+1) - D^Tu^*(t+1)] \tag{B-2}$$

from eq. (2-9)

$$x^*(t+1 - x^*(t) = AX^*(t) + BU^*(t) \tag{B-3}$$

from eq. (2-10)

$$p^*(t+1) - p^*(t) = - QX^*(t) - (1/2)SU^*(t) - A^Tp^*(t+1)$$
$$+ C^Tu^*(t+1) \tag{B-4}$$

from eq. (2-11)

$$CX^*(t) + DU^*(t) = 0 \tag{B-5}$$

from eq. (2-14)

$$p^*(N) = QX^*(N) \tag{B-6}$$

NOTE (omit * in remainder of derivation)

from (B-5)

$$DU(t) = -CX(t)$$

premultiply both sides of (B-2)

$$DU(t) = -DR^{-1}[(1/2)S^TX(t) + B^Tp(t+1) - D^Tu(t+1)]$$

thus

$$-CX(t) = -DR^{-1}[(1/2)S^TX(t) + B^Tp(t+1) - D^Tu(t+1)]$$

$$CX(t) = (1/2)DR^{-1}S^TX(t) + DR^{-1}B^Tp(t+1) - DR^{-1}D^Tu(t+1)$$

rewriting

$$DR^{-1}D^Tu(t+1) = [(1/2)DR^{-1}S^T - C]X(t) + DR^{-1}B^Tp(t+1)$$

NOTE ($DR^{-1}D^T$ = scalar, thus $(DR^{-1}D^T)^{-1}$ exists)

therefore

$$u(t+1) = (DR^{-1}D^T)^{-1}[[(1/2)DR^{-1}S^T - C]X(t)$$
$$+ DR^{-1}B^Tp(t+1)]$$

for convenience, define

$F = (DR^{-1}D^T)^{-1}$: a scalar.

thus

$$u(t+1) = [(1/2)FDR^{-1}S^T - FC]X(t) + [FDR^{-1}B^T]p(t+1) \tag{B-7}$$

substituting (B-7) into (B-2)

$$u(t) = -R^{-1}[(1/2)S^TX(t) + B^Tp(t+1) - (1/2)FD^TDR^{-1}S^TX(t)$$
$$+ FD^TCX(t) - FD^TDR^{-1}B^Tp(t+1)]$$

$$u(t) = -R^{-1}[[(1/2)S^T - (1/2)FD^TDR^{-1}S^T]X(t)$$
$$+ [B^T - FD^TDR^{-1}B^T]p(t+1)] \tag{B-8}$$

substituting (B-7) into (B-4)

$$p(t+1) - p(t) = -QX(t) - (1/2)SU(t) - A^T p(t+1) + [(1/2)FC^T DR^{-1} S^T - FC^T C]X(t) + FC^T DR^{-1} B^T p(t+1)$$

rewriting

$$p(t+1) - p(t) = [(1/2)FC^T DR^{-1} S^T - FC^T C - Q]X(t) + [FC^T DR^{-1} B^T - A^T]p(t+1) + FC^T DR^{-1} B^T p(t+1) \tag{B-9}$$

substituting (B-8) into (B-3)

$$x(t+1) - X(t) = AX(t) - BR^{-1}[[(1/2)S^T - (1/2)FD^T DR^{-1} S^T + FD^T C]X(t) + [B^T - FD^T DR^{-1} B^T]p(t+1)]$$

$$x(t+1) - X(t) = [A - (1/2)BR^{-1} S^T + (1/2)FBR^{-1} D^T DR^{-1} S^T - FBR^{-1} D^T C]X(t) + [FBR^{-1} D^T DR^{-1} B^T - BR^{-1} B^T]p(t+1) \tag{B-10}$$

substituting (B-8) into (B-9)

$$p(t+1) - p(t) = [(1/2)FC^T DR^{-1} S^T - FC^T C - Q]X(t) + [FC^T DR^{-1} B^T - A^T]p(t+1) + (1/2)SR^{-1}[[(1/2)S^T - (1/2)FD^T DR^{-1} S^T + FD^T C]X(t) + [B^T - FD^T DR^{-1} B^T]p(t+1)]$$

$$p(t+1) - p(t) = [(1/2)FC^TDR^{-1}S^T - FC^TC - Q$$
$$+ (1/4)SR^{-1}S^T - (1/4)FSR^{-1}D^TDR^{-1}S^T$$
$$+ (1/2)FSR^{-1}D^TC]X(t)$$
$$+ [FC^TDR^{-1}B^T - A^T + (1/2)SR^{-1}B^T$$
$$- (1/2)FSR^{-1}D^TDR^{-1}B^T]p(t+1) \qquad \text{(B-11)}$$

Thus we have two vector-matrix difference equations (B-10) and (B-11) subject to the split boundary conditions: $X(t=0) = X(0)$ and $p(N) = QX(N)$. Such problems are called two point boundary value problems and, in general, they are difficult to solve. Notice that equations (B-10) and (B-11) are coupled since $X(t+1)$ depends on $p(t+1)$ and $p(t+1)$ depends on $X(t)$. We assume that these variables are linearly related (this is known as the sweep method for solving a linear two point boundary problem).

Thus

$$p(t) = K(t)\ X(t) \qquad \text{(B-12)}$$

and we will see later that this will result in a unique solution.

Substituting (B-12) into (B-8)

$$U(t) = -R^{-1}[[(1/2)S^T - (1/2)FD^TDR^{-1}S^T + FD^TC]X(t)$$
$$+ [B^T - FD^TDR^{-1}B^T][K(t+1)X(t+1)]]$$

$$U(t) = [(1/2)FR^{-1}D^TDR^{-1}S^T - (1/2)R^{-1}S^T - FR^{-1}D^TC]X(t)$$

$$+ [FR^{-1}D^TDR^{-1}B^T - R^{-1}B^T]K(t+1)X(t+1) \quad \text{(B-13)}$$

substituting (B-12) into (B-10)

$$X(t+1) - X(t) = [A - (1/2)BR^{-1}S^T + (1/2)FBR^{-1}D^TDR^{-1}S^T$$

$$- FBR^{-1}D^TC]X(t) + [FBR^{-1}D^TDR^{-1}B^T$$

$$- BR^{-1}B^T]K(T+1)X(t+1) \quad \text{(B-14)}$$

rewriting

$$[I + BR^{-1}B^TK(t+1) - FBR^{-1}D^TDR^{-1}B^TK(t+1)]X(t+1)$$

$$= [I + A - (1/2)BR^{-1}S^T + (1/2)FBR^{-1}D^TDR^{-1}S^T$$

$$- FBR^{-1}D^TC]X(t) \quad \text{(B-15)}$$

substituting (B-12) into (B-11)

(RH side)

$$p(t+1) - p(t) = [(1/2)FC^TDR^{-1}S^T - FC^TC-Q+(1/4)SR^{-1}S^T$$

$$- (1/4)FSR^{-1}D^TDR^{-1}S^T$$

$$+ (1/2)FSR^{-1}D^T C]X(t)$$

$$+ [FC^TDR^{-1}B^T - A^T + (1/2)SR^{-1}B^T$$

$$- (1/2)FSR^{-1}D^TDR^{-1}B^T]K(t+1)X(t+1)$$

(LH side)

$$[I - FC^TDR^{-1}B^T + A^T - (1/2)FSR^{-1}B^T$$

$$+ (1/2)FSR^{-1}D^TDR^{-1}B^T]K(t+1)X(t+1) - K(t)X(t)$$

$$= [(1/2)FC^TDR^{-1}S^T - FC^TC - Q + (1/4)SR^{-1}S^T$$

$$- (1/4)FSR^{-1}D^TDR^{-1}S^T + (1/2)FSR^{-1}D^TC]X(t) \qquad \text{(B-16)}$$

let

$$E = [I + BR^{-1}B^TK(t+1) - FBR^{-1}D^TDR^{-1}B^TK(t+1)]$$

thus

$$X(t+1) = E^{-1}[I + A - (1/2)BR^{-1}S^T + (1/2)FBR^{-1}D^TDR^{-1}S^T$$

$$- FBR^{-1}D^TC]X(t) \qquad \text{(B-17)}$$

substituting (B-17) into (B-16)

$$[I - FC^TDR^{-1}B^T + A^T - (1/2)SR^{-1}B^T$$

$$+ (1/2)FSR^{-1}D^TDR^{-1}B^T]K(t+1)E^{-1}[I + A - (1/2)BR^{-1}S^T$$

$$+ (1/2)FBR^{-1}D^TDR^{-1}D^TC]X(t)$$

$$= [K(t) + (1/2)\ FC^TDR^{-1}S^T - FC^T - Q + (1/4)SR^{-1}S^T$$

$$- (1/4)FSR^{-1}D^TDR^{-1}S^T + (1/2)FSR^{-1}D^TC]X(t)$$

equating coefficients

$$[I - FC^TDR^{-1}B^T + A^T - (1/2)SR^{-1}B^T$$

$$+ (1/2)FSR^{-1}D^TDR^{-1}B^T]K(t+1)E^{-1}[I + A - (1/2)BR^{-1}S^T$$

$$+ (1/2)FBR^{-1}D^TDR^{-1}S^T - FBR^{-1}D^TC]$$

$$= K(t) + (1/2)FC^TDR^{-1}S^T - FC^TC - Q + (1/4)SR^{-1}S^T$$

$$- (1/4)FSR^{-1}D^TDR^{-1}S^T + (1/2)FSR^{-1}D^TC$$

let

$$H = [I + A - (1/2)BR^{-1}S^T + (1/2)FBR^{-1}D^TDR^{-1}S^T - FBR^{-1}D^TC]$$

let

$$L = [-(1/2)FSR^{-1}D^TC + (1/4)FSR^{-1}D^TDR^{-1}S^T - (1/4)SR^{-1}S^T + Q + FC^TC - (1/2)FC^TDR^{-1}S^T]$$

thus

$$K(t) = H^TK(t+1)E^{-1}H + L \qquad \text{(B-18)}$$

from (B-6) and (B-12) we have

$$p(N) = QX(N) = K(N)X(N)$$

thus

$$K(N) = Q \qquad \text{(B-19)}$$

We can solve eq. (B-18) backward to find $K(t)$, $t = 1,\ldots,N$.

Substituting (B-17) into (B-13) yields

$$U(t) = [(1/2)FR^{-1}D^TDR^{-1}S^T - (1/2)R^{-1}S^T - FR^{-1}D^TC]X(t) + [FR^{-1}D^TDR^{-1}B^T - R^{-1}B^T]K(t+1)E^{-1}HX(t)$$

for convenience, define

$$J = [(1/2)FR^{-1}D^TDR^{-1}S^T - (1/2)R^{-1}S^T - FR^{-1}D^TC]$$

$$M = [FR^{-1}D^TDR^{-1}B^T - R^{-1}B^T]$$

thus

$$U^*(t) = JX(t) + MK(t+1)E^{-1}HX(t) \quad \text{(B-20)}$$

recall

$$E = [I + BR^{-1}B^TK(t+1) - FBR^{-1}D^TDR^{-1}B^TK(t+1)]$$

using the matrix identity:

$$(I_n + ST^T)^{-1} = I_r - S(I_r + T^TS)^{-1}T^T$$

where S is (nxr), T is (nxr) and $r \leq n$

$$E = [I + B(R^{-1}B^T - FR^{-1}D^TDR^{-1}B^T)K(t+1)]$$

let

$$S = B \text{ and } T^T = (R^{-1}B^T - FR^{-1}D^TDR^{-1}B^T)K(t+1)$$

then

$$E^{-1} = I - B[I + (R^{-1}B^T - FR^{-1}D^TDR^{-1}B^T)K(t+1)B]^{-1}$$
$$(R^{-1}B^T - FR^{-1}D^TDR^{-1}B^T)K(t+1)$$

define

$$V = [B^T - FD^TDR^{-1}B^T]$$

thus

$$E^{-1} = I - B(I + R^{-1}VK(t+1)B)^{-1}R^{-1}VK(t+1)$$

$$E^{-1} = I - B[R(I + R^{-1}VK(t+1)B)]^{-1}VK(t+1)$$

$$E^{-1} = I - B[R + VK(t+1)B]^{-1}VK(t+1) \quad \text{(B-21)}$$

note that we now only need to invert an (rxr) matrix instead of an (nxn) matrix

substituting (B-21) into (B-18)

$$K(t) = H^T K(t+1)[I - B[R + VK(t+1)B]^{-1} VK(t+1)]H + L \tag{B-22}$$

substituting (B-21) into (B-20)

$$U^*(t) = JX(t) + MK(t+1)$$
$$(I - B[R + VK(t+1)B]^{-1} VK(t+1)HX(t) \tag{B-23}$$

Equation (B-23) determines the optimal control in terms of the present state and solutions of the "Ricatti" equation (B-22). Once the system has been defined (matrices A and B), the additional constraints identified (matrices C and D) and the performance measure determined (matrices Q, R, and S) the optimal decision can be found as follows:

1. Solve the Riccati equation (B-22) with boundary condition (B-19) backward in time to get K(t) for t = 1,...,N-1. Store the resulting (nxn) matrices (N of them).

2. Compute the optimal control U*(0) from equation (B-23) using the initial conditions X(t=0) = X(0) and the matrix K(1).

3. Compute the next state using equation (B-3).

4. Compute the next optimal control using equation (B-23).

5. Iterate steps (3) and (4) until all $U^*(t)$, $t = 0,\ldots,N-1$ and all $X^*(t)$ $t = 1,\ldots,N$ have been computed.

Application of the above algorithm to the model developed in Chapter II results in the following simplifications (since V and M are null matrices in the model):

$$K(t) = H^T K(t+1) H + L \tag{B-22a}$$

$$U^*(t) = JX^*(t) \tag{B-23a}$$

where H, L, and J are defined above.

APPENDIX C

OPTIMAL PLANNING POLICY FOR CENTRALIZED FIRM

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OPTIMAL PLANNING POLICY FOR CENTRALIZED FIRM

Period One

RAW MATERIAL PRICE = 20750.000
DEMAND PARAMETER (B2) = 5000.0000
INVENTORY = 0.0000000
PRODUCTION QUANTITY DECISION = 1448.0198
PRODUCT PRICING DECISION = 35519.802

Period Two

RAW MATERIAL PRICE = 21372.500
DEMAND PARAMETER (B2) = 5250.0000
INVENTORY = -2.91038305E-11
PRODUCTION QUANTITY DECISION = 1540.9653
PRODUCT PRICING DECISION = 37090.347

Period Three

RAW MATERIAL PRICE = 22013.675
DEMAND PARAMETER (B2) = 5512.5000
INVENTORY = -5.82076609E-11
PRODUCTION QUANTITY DECISION = 1639.1745
PRODUCT PRICING DECISION = 38733.255

Period Four

RAW MATERIAL PRICE = 22674.085
DEMAND PARAMETER (B2) = 5788.1250
INVENTORY = -2.91038305E-11
PRODUCTION QUANTITY DECISION = 1742.9289
PRODUCT PRICING DECISION = 40451.961

Period Five

RAW MATERIAL PRICE = 23354.308
DEMAND PARAMETER (B2) = 6077.5313
INVENTORY = -2.91038305E-11
PRODUCTION QUANTITY DECISION = 1852.5250
PRODUCT PRICING DECISION = 42250.063

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